Cluster algebras and Poisson geometry

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Plan of the talk

Part 1. Cluster algebras (by example)

The main concepts of the cluster theory via an example on GL_3.

Part 2. Cluster algebras and Poisson geometry

Compatibility between a cluster structure and a Poisson bracket; how to construct a cluster structure via Poisson geometry.

Part 3. Research program

Belavin-Drinfeld data; the three main objects of the GSV program; construction of a Poisson birational quasi-isomorphism for $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$; an explicit example in GL_3.

Part 4. Description of $\mathcal{GC}^\dagger(\mathbf{\Gamma})$ in type A

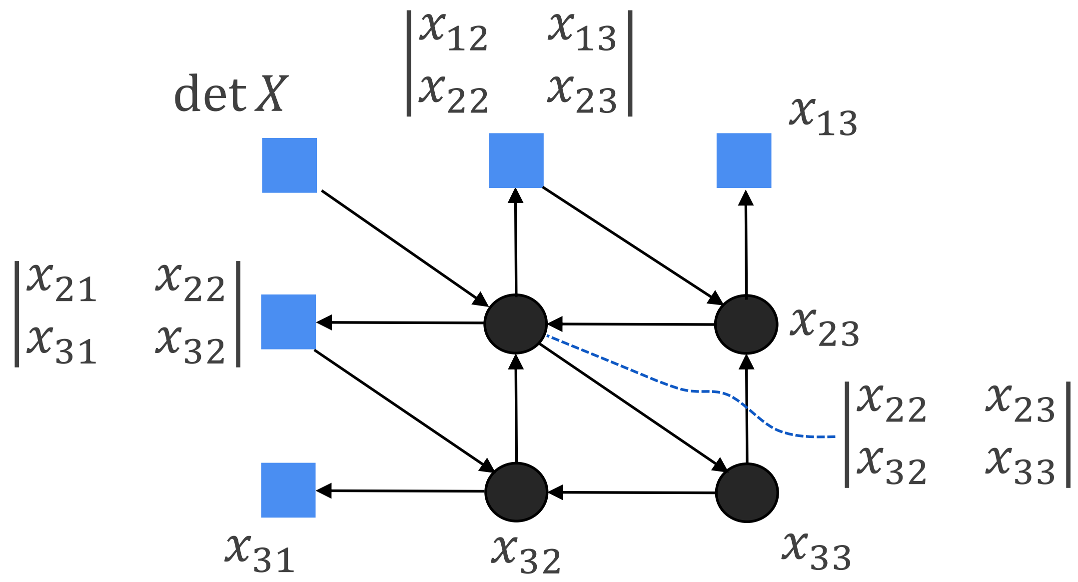
The main features of the construction and the results of the paper arXiv:2312.04859.

Part 5. Features of $\mathcal{GC}^D(\mathbf{\Gamma}^\mathrm{r}, \mathbf{\Gamma}^c)$ in type A

The relation between $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$, $\mathcal{GC}^D(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ and $\mathcal{GC}^\dagger(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$.

Part 6. Status of the program

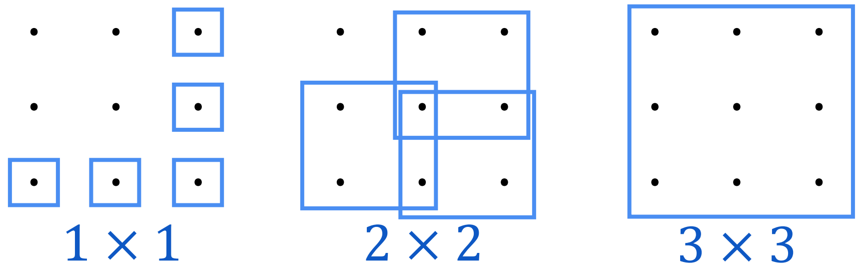
Part 1. Cluster algebras (by example)



(det is dropped for convenience)

$$X := \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathrm{GL}_3$$

The functions visualized:



(pick either the last columns or the last rows)

The (initial) extended seed:

(initial) quiver

$$\Sigma := (\boldsymbol{x}, Q)$$

(initial) extended cluster

(that is, the initial collection of functions)

frozen vertex

(the attached function is a *frozen variable*)

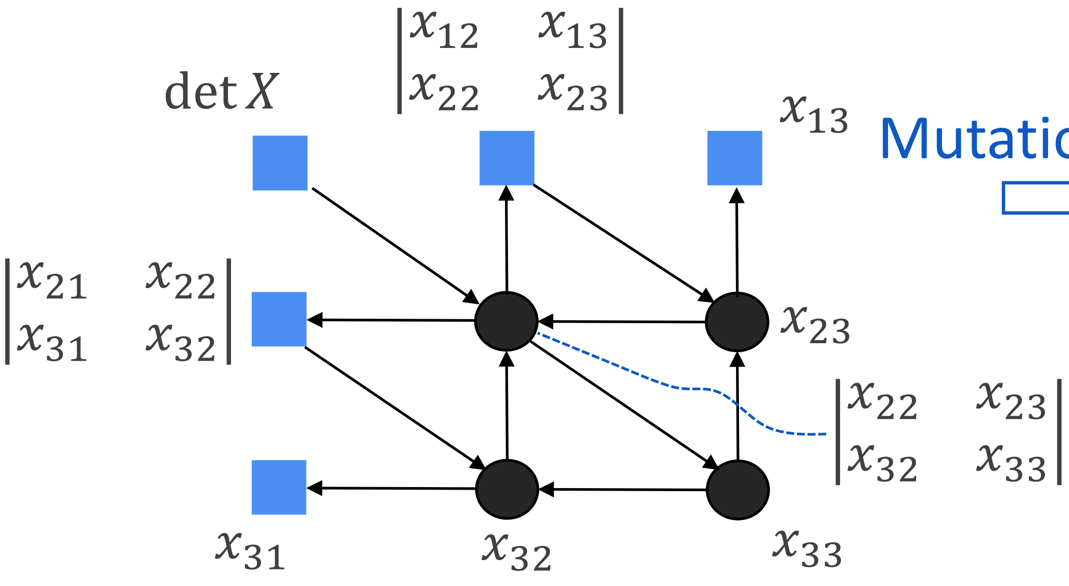
mutable vertex

(the attached function is a *cluster variable*)

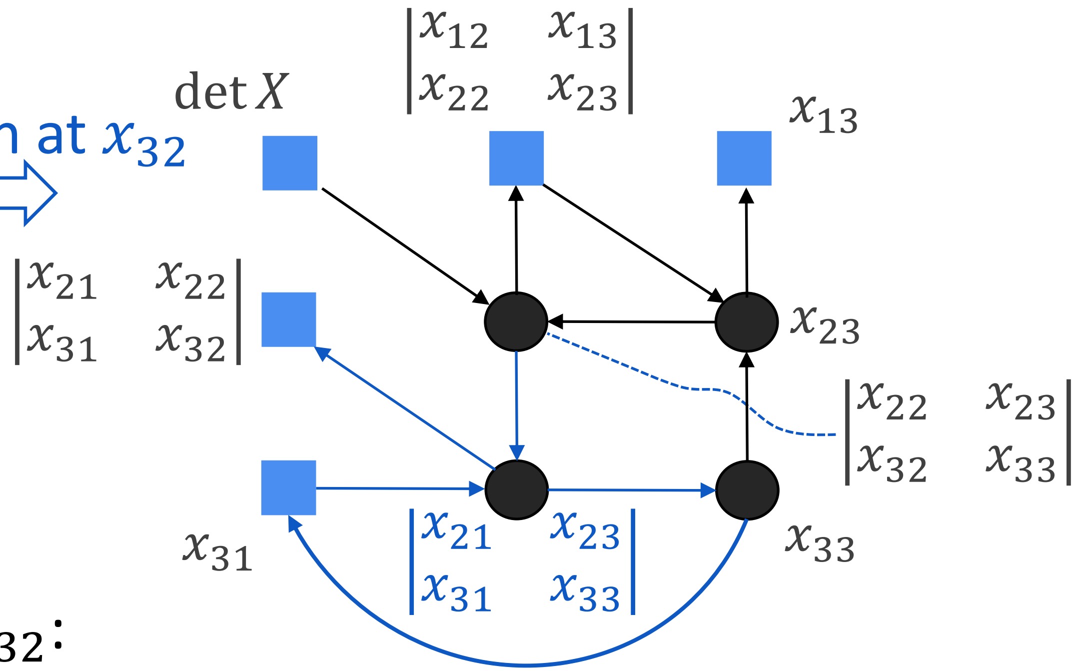
Mutation (by example)

A *mutation* is an involutive operation on (extended) seeds:

- Replaces a chosen cluster variable with a new one;
- Updates the quiver.

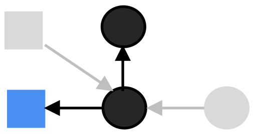


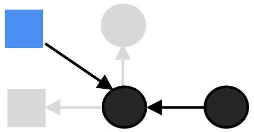
Mutation at x_{32}



A *mutation* (or *exchange*) *relation* for x_{32}:

$$x_{32} \cdot \det \begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} = \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} x_{31} + \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$





A new cluster variable that replaces x_{32}

Main definitions

Setup. $\mathbb{K}$ a field, N and M are the numbers of cluster and frozen variables in an extended cluster, respectively; Σ_0 initial extended seed.

Cluster structure $\mathcal{C} := \mathcal{C}(\Sigma_0) :=$ all possible extended seeds that can be obtained from Σ_0 via finite sequences of mutations.

Cluster algebra $\mathcal{A}_{\mathbb{K}}(\mathcal{C}) := \mathbb{K}[$ all cluster and frozen variables from $\mathcal{C}]$.

Upper cluster algebra

$$\bar{\mathcal{A}}_{\mathbb{K}}(\mathcal{C}) := \bigcap_{\text{all } \mathbf{x} \text{ in } \mathcal{C}} \mathbb{K}\Big[\overbrace{x_1^{\pm 1}, \dots, x_N^{\pm 1}}^{\substack{\text{(these vary in the intersection)} \\ \text{cluster variables}}}, \underbrace{x_{N+1}, \dots, x_{N+M}}_{\substack{\text{frozen variables} \\ \text{(these don't change)}}} \;\Big|\; x_i \in \boldsymbol{x}\Big].$$

Theorem (Laurent Phenomenon). $\mathcal{A}_{\mathbb{K}}(\mathcal{C}) \subseteq \bar{\mathcal{A}}_{\mathbb{K}}(\mathcal{C})$.

Program on cluster algebras

(S. Fomin, A. Zelevinsky, 2001)

For every interesting variety V over a field $\mathbb{K}$ in Lie theory, find $\mathcal{C}$ such that

$$\mathcal{O}(V) = \mathcal{A}_{\mathbb{K}}(\mathcal{C})$$

coordinate ring of V

Example. $\mathcal{O}(\mathrm{GL}_3) = \mathcal{A}_{\mathbb{K}}(\mathcal{C})$.

However, usually we are only able to show $\mathcal{O}(V) = \bar{\mathcal{A}}_{\mathbb{K}}(\mathcal{C})$.

(for instance, for cluster structures on double Bruhat cells $G^{u,v}$, although we know that $\mathcal{O}(G^{u,v}) = \bar{\mathcal{A}}_{\mathbb{K}}(\mathcal{C})$, we still do not know whether $\mathcal{O}(G^{u,v}) = \mathcal{A}_{\mathbb{K}}(\mathcal{C})$)

Question. How to construct a cluster structure in the first place?

Part 2. Cluster algebras and Poisson geometry

Setup. $\mathcal{A} \coloneqq$ a commutative algebra over a field $\mathbb{K}$.

Def. A *Poisson bracket* $\{\ ,\ \}: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a skew-symmetric bilinear form that satisfies the Jacobi identity and the Leibniz rule in each slot.

Jacobi identity: $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$

$\forall a, b, c \in \mathcal{A}$

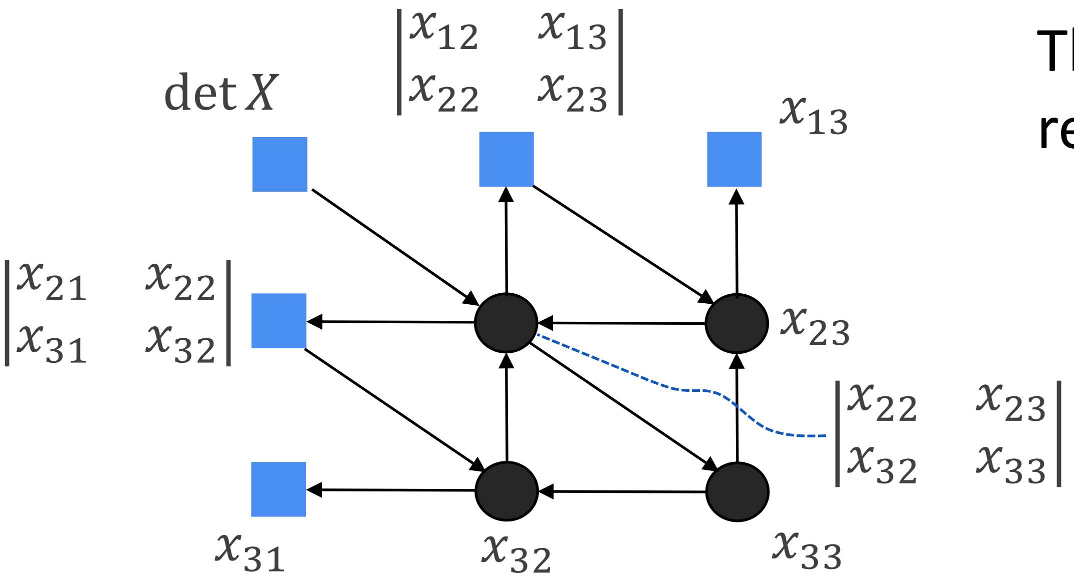
Leibniz rule: $\{a \cdot b, c\} = a\{b, c\} + \{a, c\}b$

Def. Elements $a, b \in \mathcal{A}$ are *log-canonical* if $\{a, b\} = \omega \cdot ab,\ \omega \in \mathbb{K}$.

A subset $S \subseteq \mathcal{A}$ is *log-canonical* if all elements in S are pairwise log-canonical.

Def. A cluster structure is *compatible* with a Poisson bracket if every extended cluster is log-canonical.

A remarkable observation (M. Gekhtman, M. Shapiro, A. Vainshtein, 2003)



This initial extended cluster is log-canonical with respect to the *standard* Poisson bracket $\{\,,\}_{\mathrm{std}}$ on GL_3.

(we explain later what standard means)

For example,

$$\{x_{31}, x_{32}\}_{\mathrm{std}} = x_{31}x_{32}, \quad \{x_{23}, x_{33}\}_{\mathrm{std}} = \frac{1}{2}x_{23}x_{33},$$

$$\left\{\det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}, x_{33}\right\}_{\mathrm{std}} = 0.$$

Even more is true:

- *Every* extended cluster is log-canonical;
- The frozen variables generate *Poisson prime ideals* in $\mathcal{O}(\mathrm{GL}_3)$:

$$\{(f), g\} \subseteq (f) \text{ for any frozen } f \text{ and any } g \in \mathcal{O}(\mathrm{GL}_3);$$

Geometrically, this is equivalent to saying that $\{X \in \mathrm{GL}_3 \mid f(X) = 0\}$ is a union of symplectic leaves of $\{\,,\}_{\mathrm{std}}$.

(B. Nguyen, K. Trampel, M. Yakimov, 2017)

How to verify the compatibility with a Poisson bracket?

Setup. $(\boldsymbol{x}, Q)$ the initial extended seed of some cluster structure,

quiver

$$\boldsymbol{x} := (\underbrace{x_1, \dots, x_N}_{\text{cluster variables}}, \underbrace{x_{N+1}, \dots, x_{N+M}}_{\text{frozen variables}}).$$

Exchange matrix B:

$$\text{Adjacency matrix of } Q = \begin{array}{cc} \left[\begin{array}{c:c} \multicolumn{2}{c}{B} \\ \hdashline & 0 \end{array}\right] & \begin{array}{c} N \\ M \end{array} \\ \begin{array}{cc} N & M \end{array} & \end{array}$$

Can remember $(\boldsymbol{x}, B)$ instead of $(\boldsymbol{x}, Q)$.

$\mathcal{F} := \mathbb{K}(x_1, \dots, x_{N+M})$;

$\{\,,\}: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ a Poisson bracket.

Theorem. Assume that the mutable part of Q is connected. TFAE:

i) B has full rank and the cluster structure is compatible with $\{\,,\}$;

ii) $\boldsymbol{x}$ is log-canonical and $B\Omega = [\lambda I \quad 0]$ (compatibility equation)

where $\lambda \in \mathbb{K}^*$, $\Omega := (\omega_{ij})_{i,j=1}^{N+M}$, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, $\omega_{ij} \in \mathbb{K}$.

(M. Gekhtman, M. Shapiro, A. Vainshtein, 2003)

How to construct a cluster structure?

Setup. A variety V over a field $\mathbb{K}$, $n := \dim V$.

field of rational functions on V

Empirical algorithm:

1) Introduce a Poisson bracket $\{\,,\}: \mathbb{K}(V) \times \mathbb{K}(V) \to \mathbb{K}(V)$;
2) Find a log-canonical family $\boldsymbol{x} := (x_1, \dots, x_n)$, $x_i \in \mathcal{O}(V)$;

   Set $\Omega := \left(\omega_{ij}\right)_{i,j=1}^{n}$, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, $\omega_{ij} \in \mathbb{K}$.
3) For frozen variables, choose those x_i that generate Poisson prime ideals.

   This also gives $n = N + M$ where $M =$ # frozen variables.
4) Solve the compatibility equation with respect to an $N \times (N + M)$ matrix B:

$$B\Omega = [I \;\; 0] \qquad (*)$$

(if B has entries in $\mathbb{Q}$, can multiply both sides by a common denominator)

Then $(\boldsymbol{x}, B)$ is the initial extended seed for some cluster structure C on V.

Remark. If $(*)$ has many solutions, can introduce extra structures (e.g., a toric action). This way one extracts a unique solution.

Part 3. Research program

Setup. $\mathfrak{g}$ is a simple complex Lie algebra, Π a set of simple roots,
$\mathfrak{h}$ the Cartan subalgebra, $<\,,>$ symmetric invariant nondegenerate form on $\mathfrak{g}$.

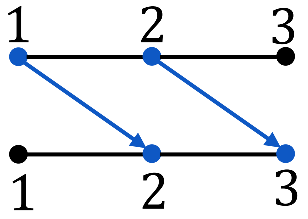
Def. A *Belavin-Drinfeld triple* (a *BD triple*) is $\mathbf{\Gamma} \coloneqq (\Gamma_1, \Gamma_2, \gamma\ )$ where $\Gamma_1, \Gamma_2 \subset \Pi$
and $\gamma\colon \Gamma_1 \to \Gamma_2$ is a nilpotent isometry.
(that is, $\forall \alpha \in \Gamma_1 \exists m > 0 : \gamma^m(\alpha) \notin \Gamma_1$)

A *Belavin-Drinfeld quadruple* (a *BD quadruple*) is $(\Gamma_1, \Gamma_2, \gamma, R_0)$ where
$R_0\colon\ \mathfrak{h} \to \mathfrak{h}$ is a linear map that satisfies

$$R_0 + R_0^* = \mathrm{id}$$
$$R_0(1-\gamma)(\alpha) = \alpha,\ \ \alpha \in \Gamma_1$$

Compatible cluster structures don't depend on R_0, but the Poisson brackets do.

Example. $\mathfrak{g} = \mathrm{sl}_4(\mathbb{C}),\ \ \Gamma_1 \coloneqq \{1,2\},\ \ \Gamma_2 \coloneqq \{2,3\},\ \ \gamma(1) \coloneqq 2,\ \ \gamma(2) \coloneqq 3.$



γ is an *isometry*: $\langle 1,2\rangle = \langle 2,3\rangle = \langle \gamma(1), \gamma(2)\rangle$

γ is *nilpotent*: $1 \overset{\gamma}{\mapsto} 2 \overset{\gamma}{\mapsto} 3 \notin \Gamma_1$

Setup. G is a connected simply connected simple complex algebraic group.

(A *BD pair*; that is, both $\boldsymbol{\Gamma}^r$ and $\boldsymbol{\Gamma}^c$ are BD triples)

$(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)$ $\Rightarrow$ Belavin & Drinfeld classification (1982) $\Rightarrow$

(classification of factorizable quasi-triangular Poisson brackets)

$\left(G, \{\,,\}_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)}\right)$ the group itself $\Rightarrow$ $\mathcal{GC}(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)$

$\left(D(G), \{\,,\}^D_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)}\right)$ $D(G) \coloneqq G \times G$ as a group; it is the *Drinfeld double* of G. $\Rightarrow$ $\mathcal{GC}^D(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)$

$\left(G, \{\,,\}^\dagger_{\boldsymbol{\Gamma}^c}\right)$ A realization of the *dual Poisson bracket* in G $\Rightarrow$ $\mathcal{GC}^\dagger(\boldsymbol{\Gamma}^c)$

The following maps are Poisson:

$$\left(G, \{\,,\}_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)}\right) \ni g \mapsto (g, g) \in \left(D(G), \{\,,\}^D_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)}\right);$$

$$\left(D(G), \{\,,\}^D_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)}\right) \ni (g, h) \mapsto g^{-1}h \in \left(G, \{\,,\}^\dagger_{\boldsymbol{\Gamma}^c}\right).$$

(at least they show relations between the three objects)

Research: Construct compatible *generalized cluster structures* $\mathcal{GC}$.

We also require:

$$\mathcal{O}(G) \text{ or } \mathcal{O}\big(D(G)\big) = \bar{\mathcal{A}}_{\mathbb{C}}(\mathcal{GC})$$

(Initiated by M. Gekhtman, M. Shapiro and A. Vainshtein in 2010)

Sidenote: An extension of $\gamma\colon \Gamma_1 \to \Gamma_2$

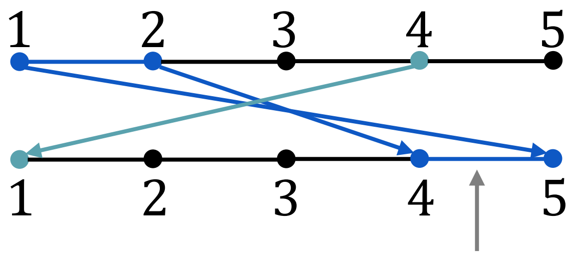
Setup. G is a connected simply connected simple complex algebraic group,
$\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$,
$\mathcal{N}_\pm, \mathcal{H}$ connected subgroups of G that correspond to $\mathfrak{n}_\pm$ and $\mathfrak{h}$,
$\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ a BD triple.

$\gamma\colon \Gamma_1 \to \Gamma_2$ can be extended to a linear map $\gamma\colon \mathfrak{g} \to \mathfrak{g}$ (a precise construction is in the papers)

The restrictions $\gamma|_{\mathfrak{n}_\pm}\colon \mathfrak{n}_\pm \to \mathfrak{n}_\pm$ are Lie algebra homomorphisms (the same for $\gamma^*\colon \Gamma_2 \to \Gamma_1$)

$\Longrightarrow$ Get Lie group homomorphisms $\tilde{\gamma}\colon \mathcal{N}_\pm \to \mathcal{N}_\pm$

Example. $\mathfrak{g} = \mathrm{gl}_6(\mathbb{C})$, $\mathbf{\Gamma}$ as on the graph.



γ is decreasing on this interval, so get a twist by W_0

$$\gamma \begin{bmatrix} X^{[1,3]}_{[1,3]} & * & * \\ * & X^{[4,5]}_{[4,5]} & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} X^{[4,5]}_{[4,5]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_0\left(-\left[X^{[1,3]}_{[1,3]}\right]^t\right)W_0^{-1} \end{bmatrix}$$

$(X \in \mathrm{gl}_6(\mathbb{C}))$

Longest Weyl group element

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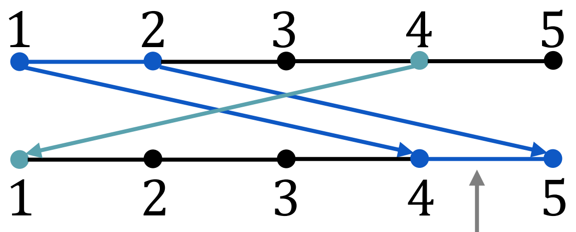

γ is increasing on this interval, so no twist by W_0

$$\gamma \begin{bmatrix} X_{[1,3]}^{[1,3]} & * & * \\ * & X_{[4,5]}^{[4,5]} & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} X_{[4,5]}^{[4,5]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X_{[1,3]}^{[1,3]} \end{bmatrix}$$

$(X \in \mathrm{gl}_6(\mathbb{C}))$

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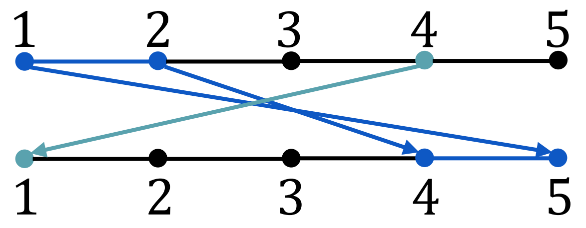


$$\tilde{\gamma}\begin{bmatrix} N_{+\,[1,3]}^{\,[1,3]} & * & * \\ 0 & N_{+\,[4,5]}^{\,[4,5]} & * \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} N_{+\,[4,5]}^{\,[4,5]} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & W_0 \left[N_{+\,[1,3]}^{\,[1,3]}\right]^{-t} W_0^{-1} \end{bmatrix}$$

$(N_+ \in \mathcal{N}_+)$

Sidenote: The R-matrix

The main ingredient for $\{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$, $\{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$ and $\{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$ is a pair of *factorizable quasi-triangular R-matrices $(R_{\mathbf{\Gamma}^r}, R_{\mathbf{\Gamma}^c})$.*

(they satisfy the *Classical Yang-Baxter equation and depend on the BD quadruples*)

For a *trivial* BD quadruple $(\emptyset, \emptyset, \emptyset, R_0)$, $R \coloneqq R_{\text{std}}$ is given by

$$R_{\text{std}} = \pi_> + R_0 \pi_0$$

projection onto $\mathfrak{h}$

projection onto $\mathfrak{n}_+$

For a *nontrivial* BD quadruple $(\Gamma_1, \Gamma_2, \gamma, R_0)$, $R \coloneqq R_{\mathbf{\Gamma}}$ is given by

$$R_{\mathbf{\Gamma}} = R_{\text{std}} + \frac{\gamma}{1-\gamma}\pi_> - \frac{\gamma^*}{1-\gamma^*}\pi_<$$

Denote $\{\,,\}_{\text{std}}$, $\{\,,\}^D_{\text{std}}$ and $\{\,,\}^D_{\text{std}}$ for $(\mathbf{\Gamma}_{\text{std}}, \mathbf{\Gamma}_{\text{std}})$, $\mathbf{\Gamma}_{\text{std}} \coloneqq (\emptyset, \emptyset, \emptyset)$.

(in a sense, a nontrivial BD data *deforms* a standard Poisson bracket; we get extra terms)

Case of $(\mathrm{GL}_n, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)})$ and $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$

Define

$$\langle A, B\rangle := \mathrm{tr}(A \cdot B),\ \ A, B \in \mathrm{gl}_n; \qquad \nabla_X f \ := \left(\frac{\partial f}{\partial x_{ji}}\right)_{i,j=1}^{n}, \qquad f \in \mathcal{O}(\mathrm{GL}_n);$$

The Poisson bracket $\{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$:

$$\{f, g\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}(X) := \langle R_{\mathbf{\Gamma}^c}(\nabla_X f \cdot X), \nabla_X g \cdot X\rangle - \langle R_{\mathbf{\Gamma}^r}(X\nabla_X f), X\nabla_X g\rangle, \qquad X \in \mathrm{GL}_n.$$

Question

A BD pair $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ deforms $\{\,,\}_{\mathrm{std}}$ into $\{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$.
How to deform the initial extended cluster of $\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$ to derive an initial extended cluster for $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$?

Answer

Construct a Poisson birational map $\mathcal{U}: (G, \{\,,\}_{\mathrm{std}}) \dashrightarrow (G, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)})$.
Then apply $(\mathcal{U}^{-1})^*$ to the initial extended cluster of $\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$.

(the same idea for $\mathcal{GC}^D(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ and $\mathcal{GC}^\dagger(\mathbf{\Gamma}^c)$)

The map $\mathcal{U}$ for $(\mathrm{GL}_n, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)})$

Gauss decomposition:

For a generic $X \in \mathrm{GL}_n$, write $X = X_+ X_0 X_-$ where $X_\pm \in \mathcal{N}_\pm$ and $X_0 \in \mathcal{H}$.

Define the maps $\rho_r\colon \mathcal{N}_+ \to \mathcal{N}_+$ and $\rho_c^*\colon \mathcal{N}_- \to \mathcal{N}_-$:

$$\rho_r(N_+) \coloneqq \overleftarrow{\prod_{i\geq 1}} (\tilde{\gamma}_r)^i(N_+), \quad \rho_c^*(N_-) \coloneqq \overrightarrow{\prod_{j\geq 1}} (\tilde{\gamma}_c^*)^j(N_-), \qquad N_\pm \in \mathcal{N}_\pm.$$

The map $\mathcal{U}\colon (G, \{\,,\}_{\mathrm{std}}) \dashrightarrow (G, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)})$:

$$\mathcal{U}(X) \coloneqq \rho_r([XW_0]_+) \cdot X \cdot \rho_c^*([W_0X]_-)$$

Theorem.

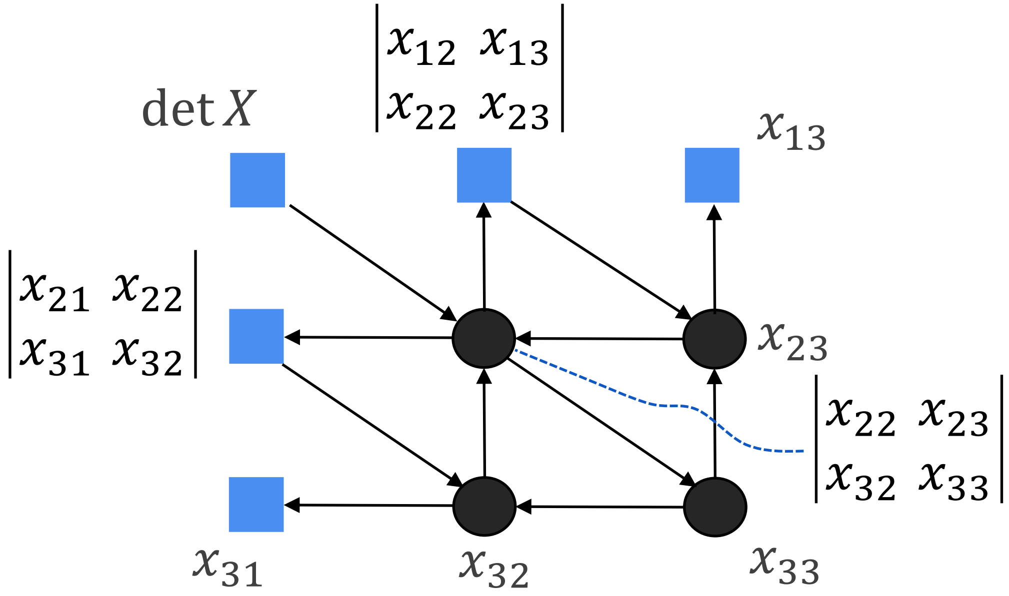
1) If (R_0^r, R_0^c) are the same for both $\{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$ and $\{\,,\}_{\mathrm{std}}$, the map is Poisson;

2) $\mathcal{U}^{-1}$ is rational if and only if $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ is *aperiodic*.

(that is, $\gamma_r w_0 \gamma_c^* w_0^{-1}$ is nilpotent)

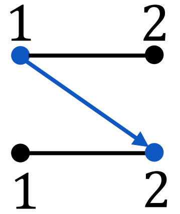
(Gekhtman-Shapiro-Vainshtein described $\mathcal{U}$ via Berenstein-Fomin-Zelevinsky parameters in their paper (2023); I found later that there is a closed formula)

An explicit example in GL_3



$\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

Set $\mathbf{\Gamma} := \mathbf{\Gamma}^r := \mathbf{\Gamma}^c := (\{1\}, \{2\}, 1 \mapsto 2)$.



Then $\mathcal{U}$ is given by

$$\mathcal{U}(X) = \left[I + \frac{\det \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}}{\det \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}} e_{23} \right] \cdot X \cdot \left[I + \frac{x_{12}}{x_{13}} e_{21} \right]$$

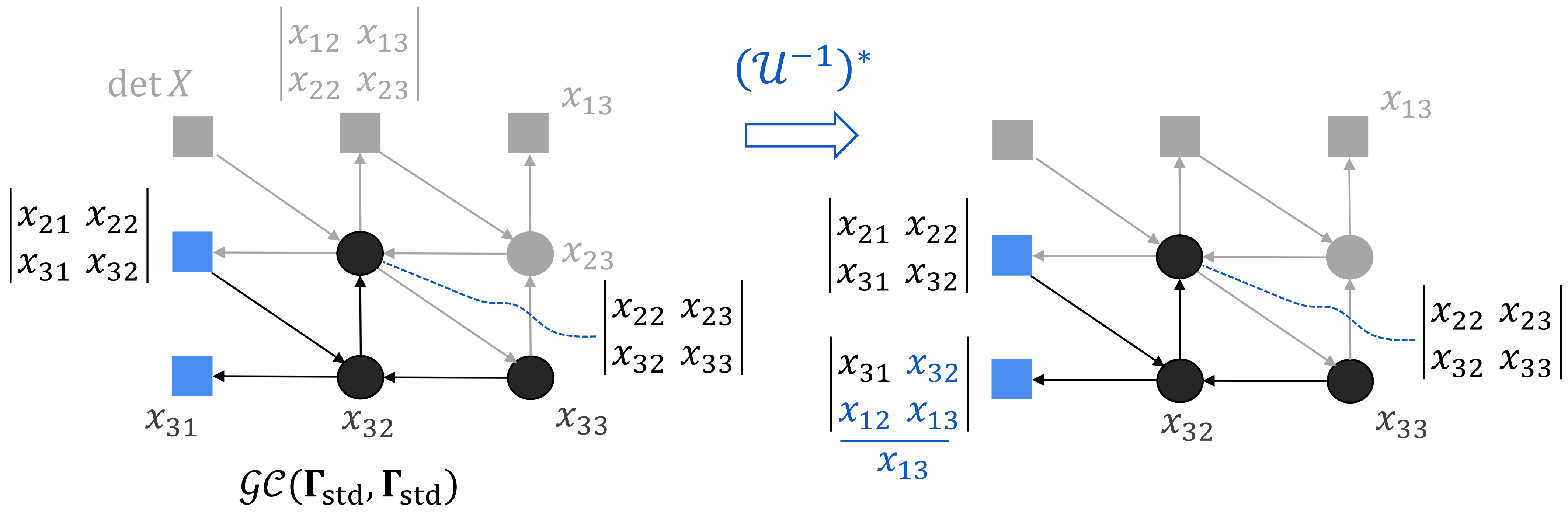
Frozen in $\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

Nuance. An application of $(\mathcal{U}^{-1})^*$ yields rational functions.

However, cluster and frozen variables have to be regular functions.

Thus we extract only the numerators. But how does that affect the quiver?

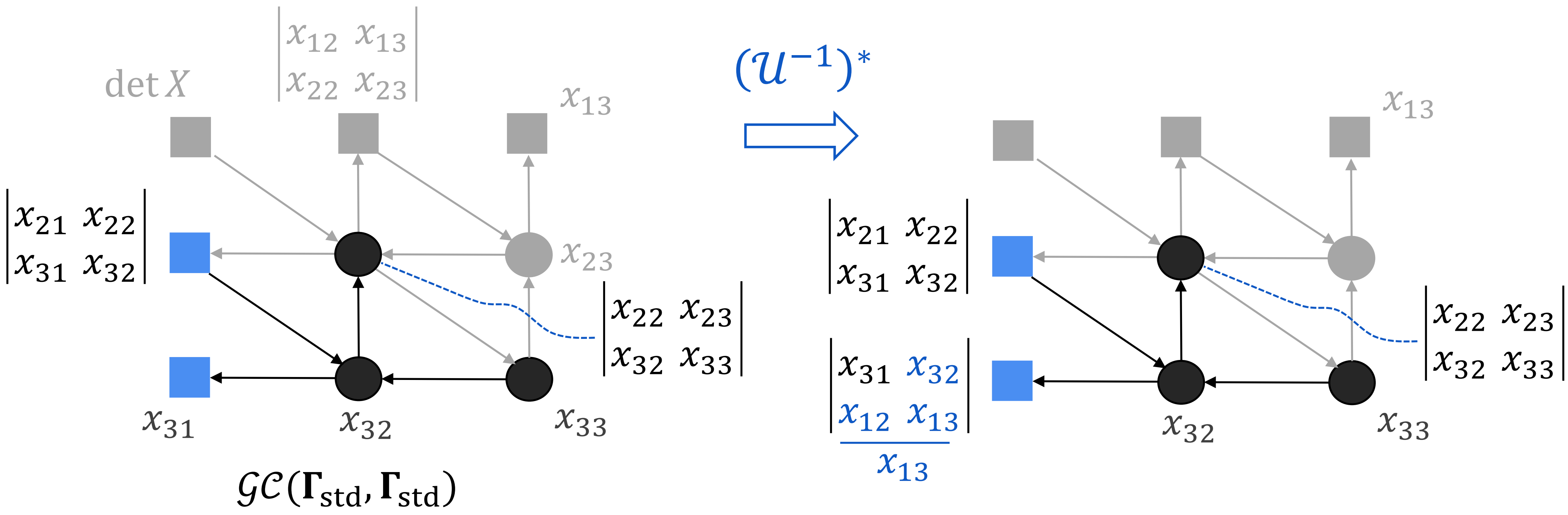
Changes in the quiver



A mutation relation for x_{32} on the left:

$$x_{32} \cdot \det \begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} = \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} x_{31} + \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$

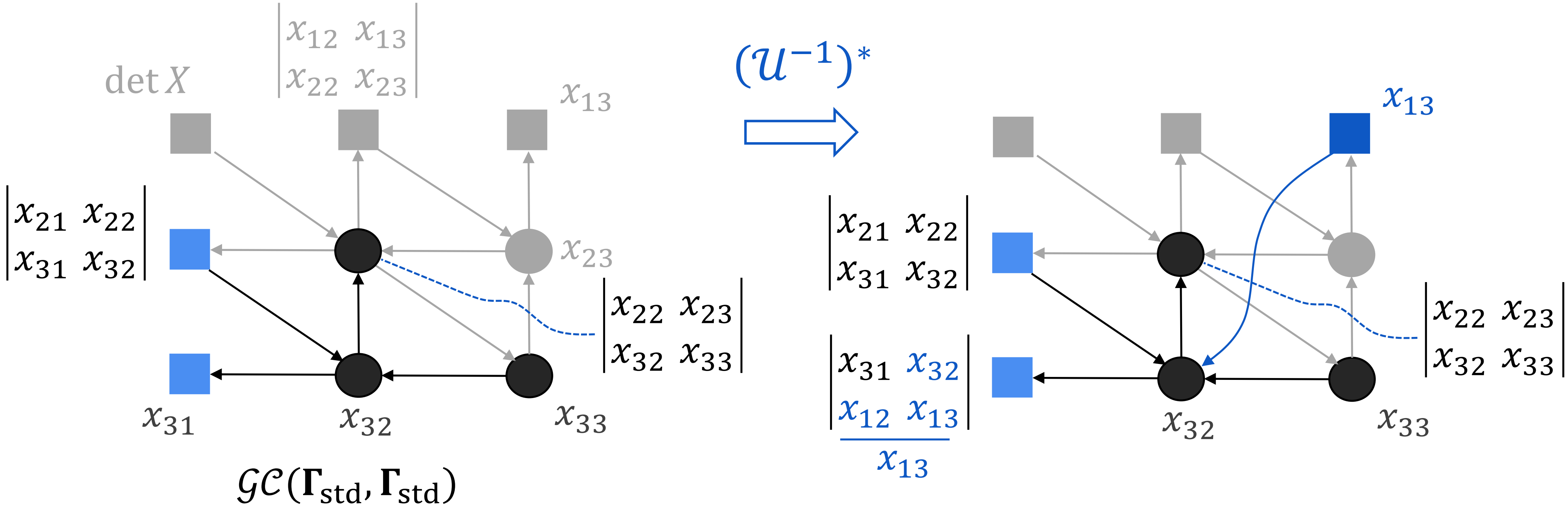
Changes in the quiver



An application of $(\mathcal{U}^{-1})^*$ to the mutation relation yields:

$$x_{32} \cdot \frac{\det \begin{bmatrix} x_{13} & x_{12} & 0 \\ x_{22} & x_{21} & x_{23} \\ x_{32} & x_{31} & x_{33} \end{bmatrix}}{x_{13}} = \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} \frac{\det \begin{bmatrix} x_{31} & x_{32} \\ x_{12} & x_{13} \end{bmatrix}}{x_{13}} + \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$

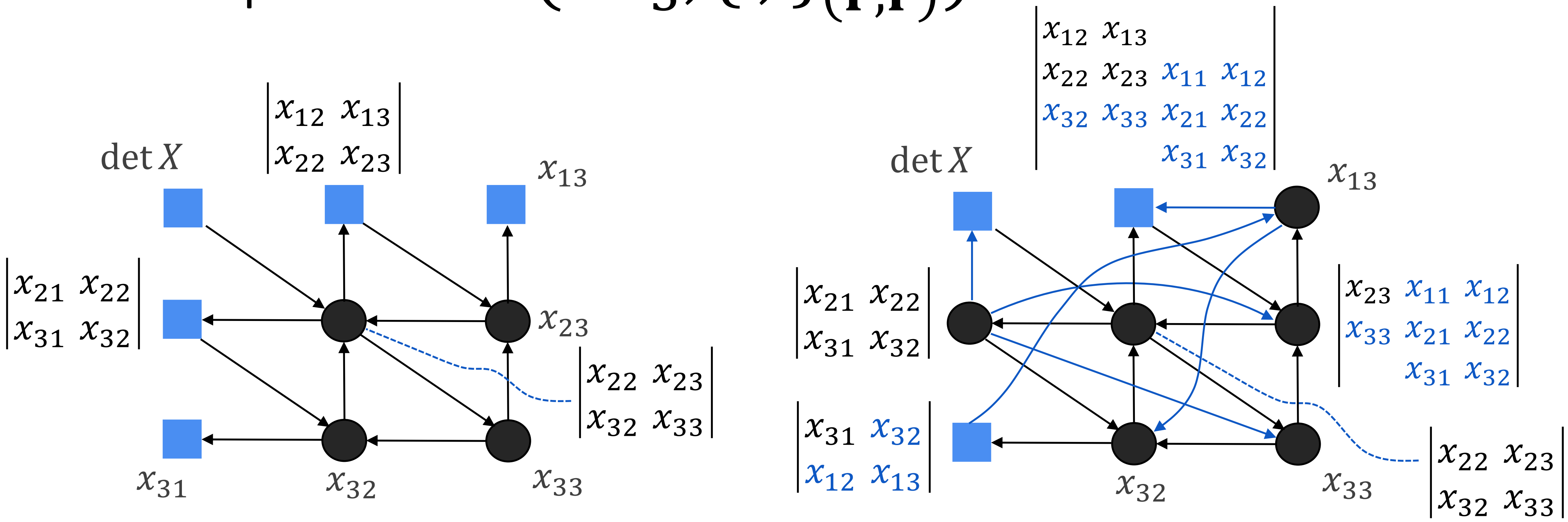
Changes in the quiver



Once we multiply by x_{13}, we get a mutation relation for x_{32} on the right. Hence an arrow!

$$x_{32} \cdot \det \begin{bmatrix} x_{13} & x_{12} & 0 \\ x_{22} & x_{21} & x_{23} \\ x_{32} & x_{31} & x_{33} \end{bmatrix} = \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} \det \begin{bmatrix} x_{31} & x_{32} \\ x_{12} & x_{13} \end{bmatrix} + \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33} x_{13}$$

Final quiver for $(\mathrm{GL}_3, \{\ ,\ \}_{(\boldsymbol{\Gamma},\boldsymbol{\Gamma})})$



$\mathcal{GC}(\boldsymbol{\Gamma}_{\mathrm{std}}, \boldsymbol{\Gamma}_{\mathrm{std}})$

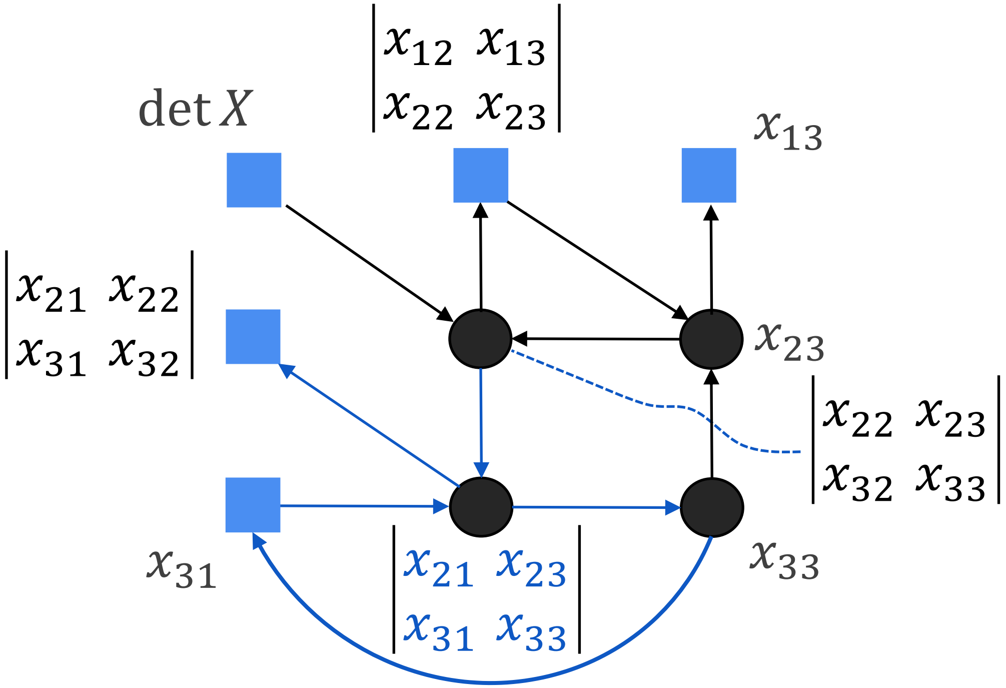
$\mathcal{GC}(\boldsymbol{\Gamma}, \boldsymbol{\Gamma})$ with $\boldsymbol{\Gamma} = (\{1\}, \{2\}, 1 \mapsto 2)$

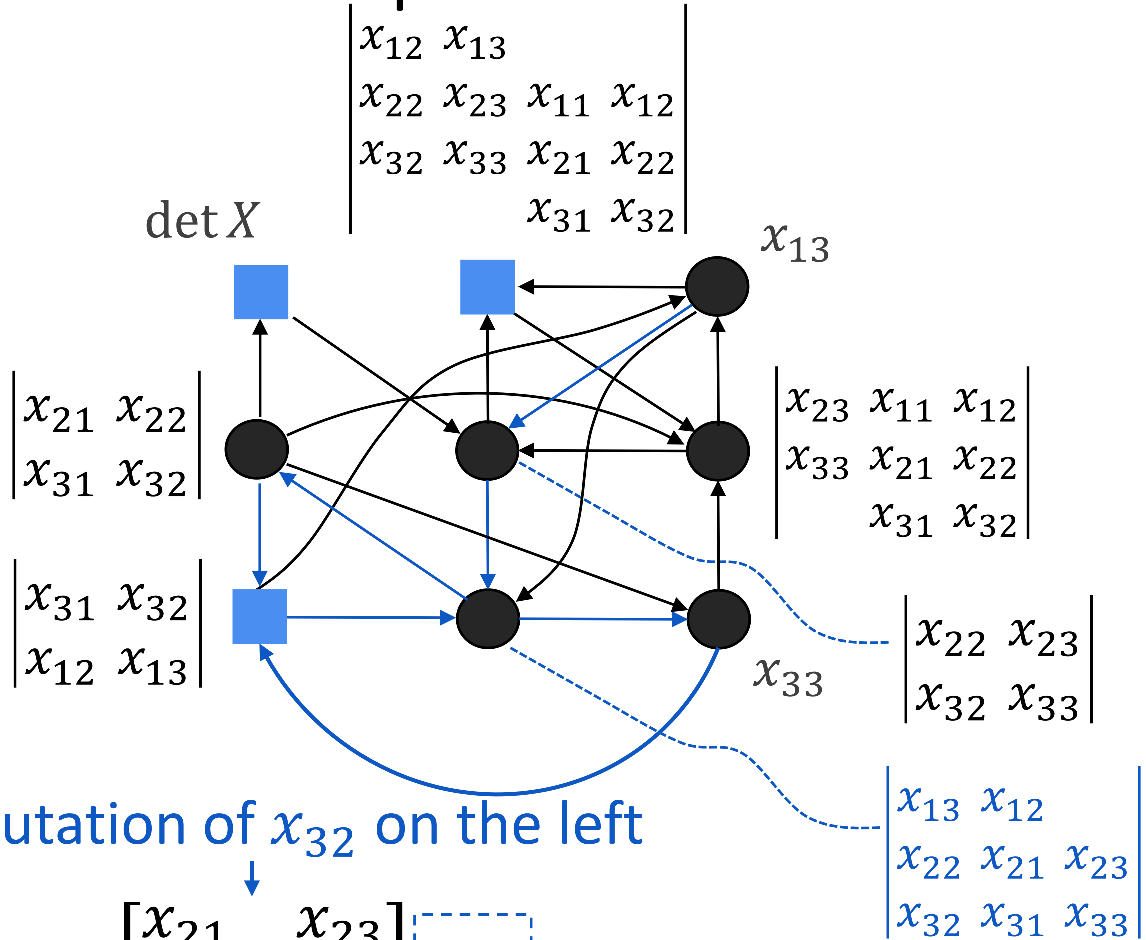
Note: we get fewer frozen variables on the right.

The variables x_{13} and $\det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$ are *marked*: they are frozen on the left but not frozen on the right. And it is precisely those variables that appear in the denominators of $\mathcal{U}$. It is always the case.

Final feature: $\mathcal{U}$ is a quasi-isomorphism

Mutate both seeds at x_{32}, for instance:





Mutation of x_{32} on the left

$$\mathcal{U}^*\left(\det\begin{bmatrix} x_{13} & x_{12} & 0 \\ x_{22} & x_{21} & x_{23} \\ x_{32} & x_{31} & x_{33} \end{bmatrix}\right) = \det\begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} \boxed{x_{13}}$$

Mutation of x_{32} on the right

In general, a monomial in the marked variables

Part 4. Description of $\mathcal{GC}^{\dagger}(\mathbf{\Gamma})$ in type A

The contrast between $\{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}$ and $\{\,,\}^{\dagger}_{\mathbf{\Gamma}^c}$:

The trace form

$$\{f,g\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}(U) = \langle R_{\mathbf{\Gamma}^c}(\nabla_U f \cdot U), \nabla_U g \cdot U\rangle - \langle R_{\mathbf{\Gamma}^r}(U\nabla_U f), U\nabla_U g\rangle, \qquad U \in \mathrm{GL}_n;$$

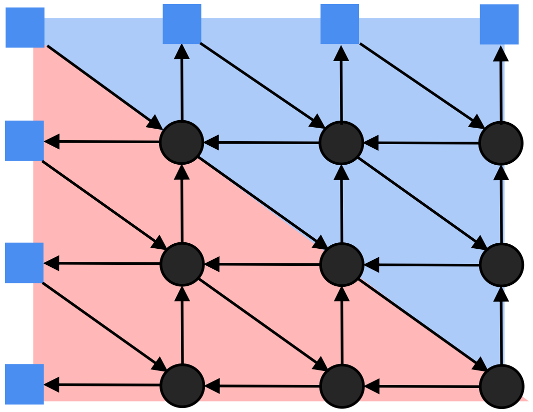
Matrix of partial derivatives of f

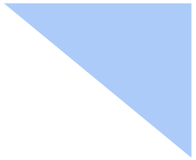
$$\{f,g\}^{\dagger}_{\mathbf{\Gamma}^c}(U) = \langle R_{\mathbf{\Gamma}^c}([\nabla_U f, U]), [\nabla_U g, U]\rangle - \langle [\nabla_U f, U], \nabla_U g U\rangle.$$

(at least one can say that conjugation plays a certain role in $\mathcal{GC}^{\dagger}(\mathbf{\Gamma})$)

Recall $\mathcal{GC}(\mathbf{\Gamma}^r_{\mathrm{std}}, \mathbf{\Gamma}^c_{\mathrm{std}})$ for $(\mathrm{GL}_n, \{\ ,\ \}_{\mathrm{std}})$

The initial extended cluster consists of two sets of flag minors.





Flag minors with last columns:

u_{14}, $\det\begin{bmatrix} u_{23} & u_{24} \\ u_{33} & u_{34} \end{bmatrix}$, etc.

Flag minors with last rows:

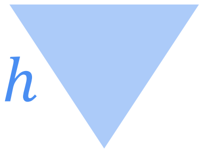
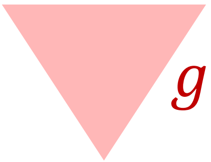
u_{42}, $\det\begin{bmatrix} u_{31} & u_{32} \\ u_{41} & u_{42} \end{bmatrix}$, etc.

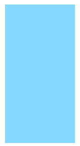
$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \in \mathrm{GL}_4$$

The initial extended cluster for $\mathcal{GC}^{\dagger}(\boldsymbol{\Gamma}_{\mathrm{std}})$ in type A

Select only one set of flag minors.

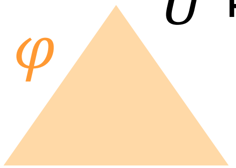
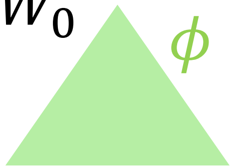
(hence, two versions of $\mathcal{GC}^{\dagger}(\boldsymbol{\Gamma})$)

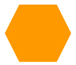
h  $U \mapsto U^T$  g

c  $c_i(U) := \chi_{\Lambda^i \mathbb{R}^n}(U)$

(characters; also, Casimirs of $\{\,,\}^{\dagger}_{\boldsymbol{\Gamma}^c}$)

$U \mapsto W_0^{-1} U W_0$

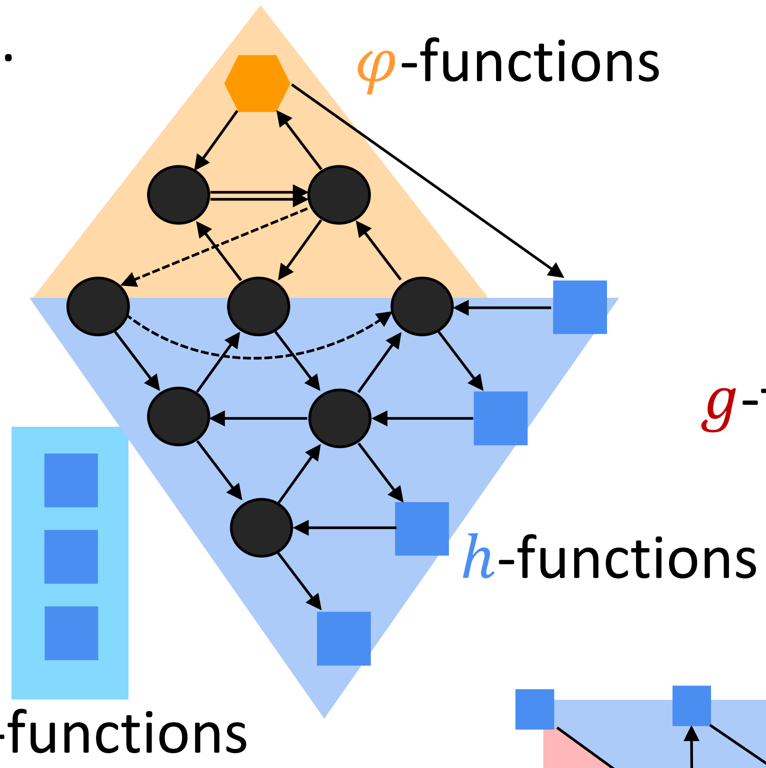
φ   ϕ

 a vertex with a generalized mutation

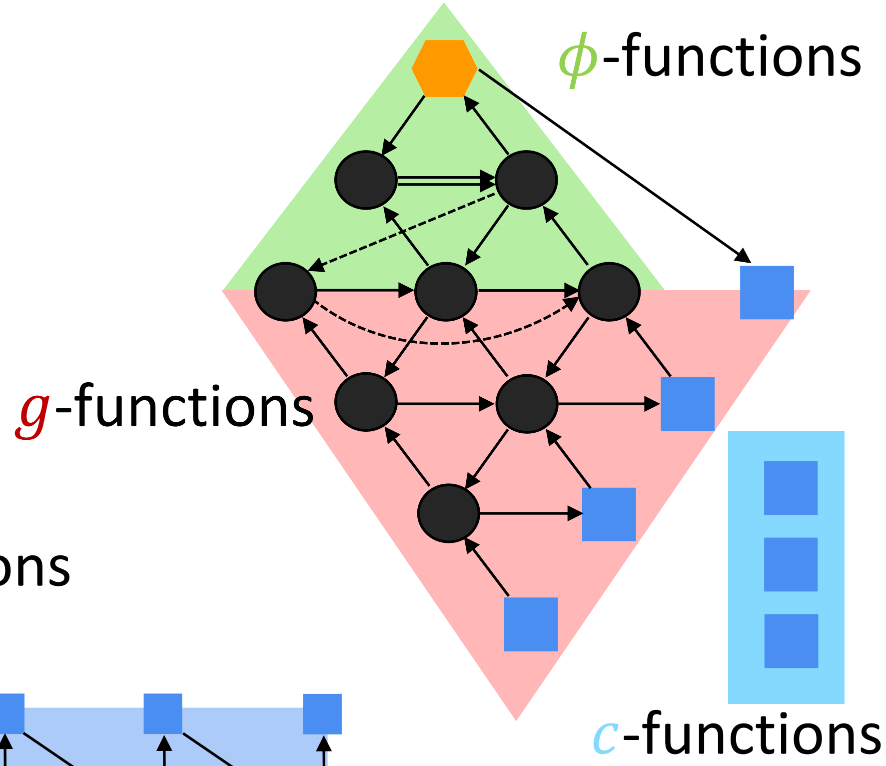
$$\varphi_{11}(U) = \det[e_n \quad Ue_n \quad \cdots \quad U^{n-1}e_n]$$

$$\phi_{11}(U) = \det[e_1 \quad Ue_1 \quad \cdots \quad U^{n-1}e_1]$$

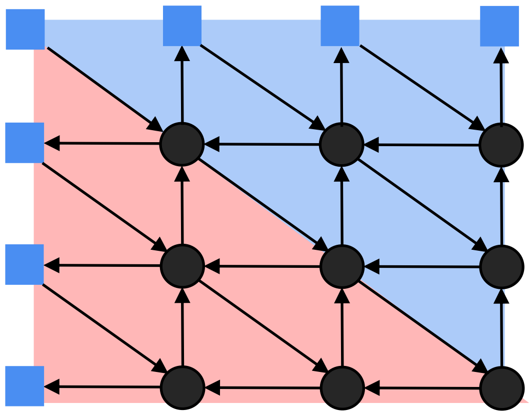
(related to the highest weight theory)



$\mathcal{GC}_h^{\dagger}(\boldsymbol{\Gamma}_{\mathrm{std}})$

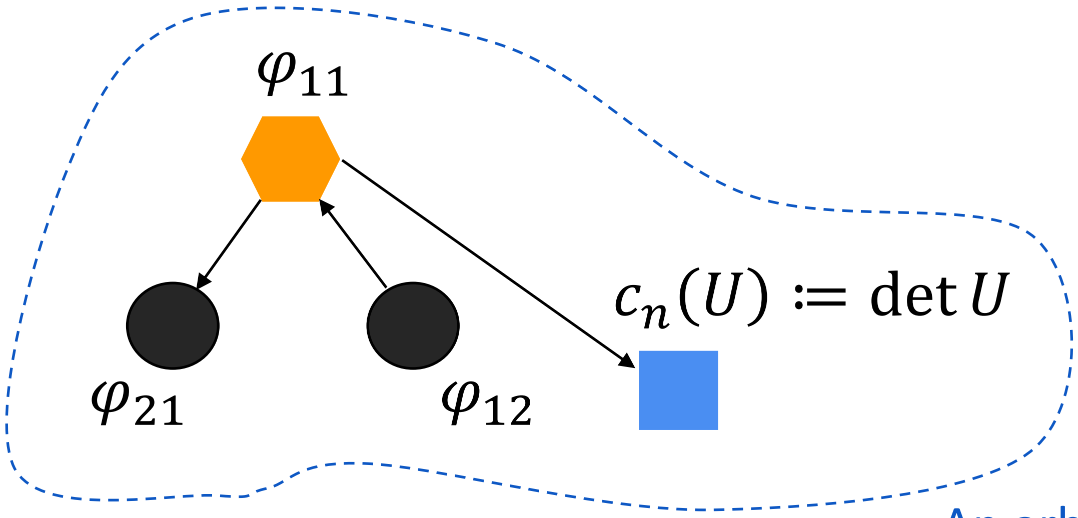


$\mathcal{GC}_g^{\dagger}(\boldsymbol{\Gamma}_{\mathrm{std}})$



$\mathcal{GC}(\boldsymbol{\Gamma}_{\mathrm{std}}, \boldsymbol{\Gamma}_{\mathrm{std}})$

A closer look at the generalized mutation



An arbitrary number of terms in the RHS is allowed

$$\varphi_{11}\varphi'_{11} = \sum_{r=0}^{\boxed{n}} \boxed{c_r}\,\varphi_{21}^{r}\,\varphi_{12}^{n-r}$$

Coefficients are allowed, but only Casimirs.

(in this case, $\{c_i \mid i \in [1, n-1]\}$ serve both as coefficients and isolated frozen variables; the variable c_n is frozen but not isolated; $c_0 = 1$)

Poisson birational quasi-isomorphisms for the dagger

Setup. G is a reductive complex Lie group, $\mathbf{\Gamma} \coloneqq (\Gamma_1, \Gamma_2, \gamma)$ any BD triple.

Gauss decomposition:

For a generic $U \in G$, write $U = U_+ U_0 U_-$ where $U_\pm \in \mathcal{N}_\pm$ and $U_0 \in \mathcal{H}$.

Recall the map $\rho^*\colon \mathcal{N}_- \to \mathcal{N}_-$:

$$\rho^*(N_-) \coloneqq \overrightarrow{\prod_{j\geq 1}} (\tilde{\gamma}^*)^j(N_-), \qquad N_\pm \in \mathcal{N}_\pm.$$

The map $\mathcal{Q}_h\colon (G, \{\,,\}^\dagger_{\text{std}}) \dashrightarrow (G, \{\,,\}^\dagger_{\mathbf{\Gamma}})$ (h-convention):

$$\mathcal{Q}_h(U) \coloneqq \rho^*(U_-)^{-1} U \rho^*(U_-), \quad U \in G.$$

(it's similar for the g-convention, see the end of the slides or the paper)

The rational inverse of $\mathcal{Q}_h$:

$$\mathcal{Q}_h^{-1}(U) \coloneqq \mathcal{F}_h(U) \cdot \tilde{\gamma}^*([\mathcal{F}_h(U)]_-)^{-1}$$

where the birational map $\mathcal{F}_h\colon G \dashrightarrow G$ satisfies the equation

$$\mathcal{F}_h(U) = \tilde{\gamma}^*([\mathcal{F}_h(U)]_-) U$$

(the equation can be solved via a recursive procedure)

Poisson birational quasi-isomorphisms for the dagger

Setup. G is a reductive complex Lie group, $\mathbf{\Gamma} \coloneqq (\Gamma_1, \Gamma_2, \gamma)$ any BD triple.

Gauss decomposition:

For a generic $U \in G$, write $U = U_+ U_0 U_-$ where $U_\pm \in \mathcal{N}_\pm$ and $U_0 \in \mathcal{H}$.

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$$\mathcal{Q}_h(U) \coloneqq \rho^*(U_-)^{-1} U \rho^*(U_-), \quad U \in G.$$

(it's similar for the g-convention, see the end of the slides or the paper)

Theorem

1) If the R_0 parts of $\{\,,\}_{\text{std}}^\dagger$ and $\{\,,\}_{\mathbf{\Gamma}}^\dagger$ are the same, then $\mathcal{Q}_h$ is Poisson;

2) If $G \in \{\mathrm{SL}_n, \mathrm{GL}_n\}$, then $\mathcal{Q}_h$ is a quasi-isomorphism between $\mathcal{GC}^\dagger(\mathbf{\Gamma}_{\text{std}})$ and $\mathcal{GC}^\dagger(\mathbf{\Gamma})$.

(D. Voloshyn, M. Gekhtman, 2023)

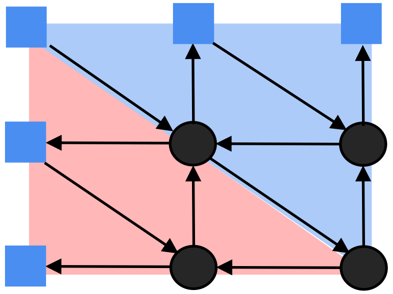
Part 5. Features of $\mathcal{GC}^D(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ in type A

Relation between the three Poisson varieties:

$$\left(\mathrm{GL}_n, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right) \ni X \mapsto (X, X) \in \left(D(\mathrm{GL}_n), \{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right);$$

$$\left(D(\mathrm{GL}_n), \{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right) \ni (X, Y) \mapsto U := X^{-1}Y \in \left(\mathrm{GL}_n, \{\,,\}^\dagger_{\mathbf{\Gamma}^c}\right).$$

$D(\mathrm{GL}_n) := \mathrm{GL}_n \times \mathrm{GL}_n$ (as a Lie group)

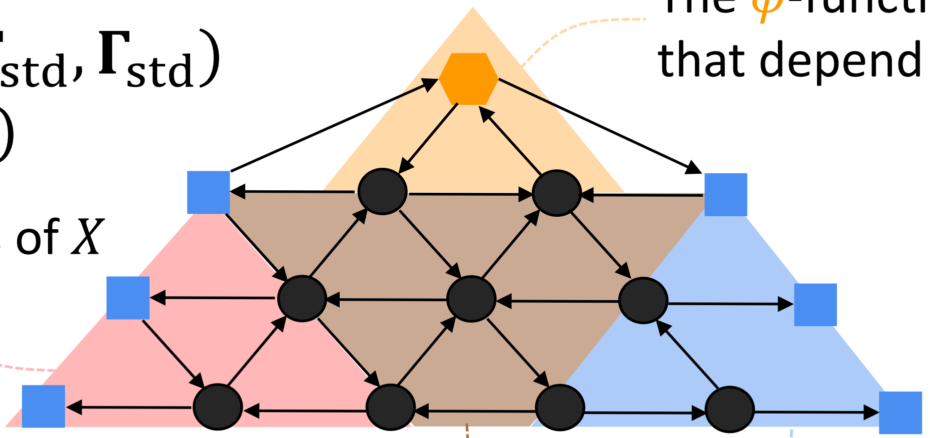


$\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

GL_3

$\mathcal{GC}^D(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

$\mathrm{D}(\mathrm{GL}_3)$

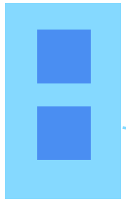


The φ-functions that depend on U

Flag minors of X

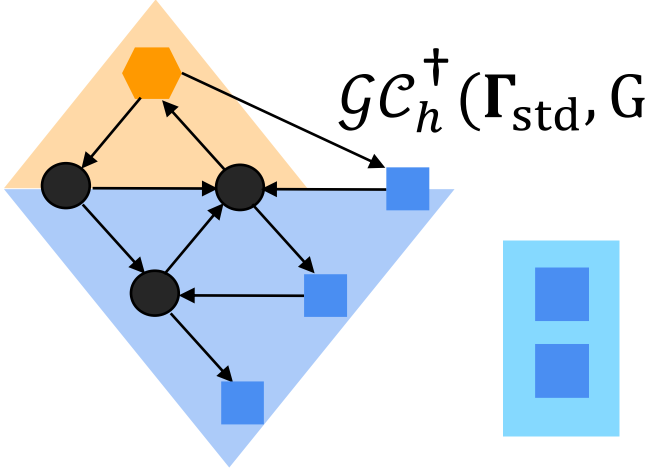
Minors of both X and Y

Flag minors of Y

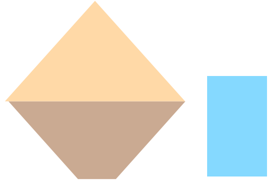


The c-functions (they depend on U)

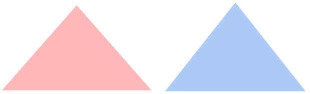
Coordinates $(X, Y) \in D(\mathrm{GL}_n)$



$\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_3)$



Stay the same for all $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$



Depend on $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$

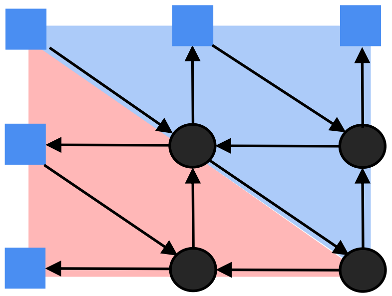
How to see $\mathcal{GC}$ inside $\mathcal{GC}^D$

Relation between the three Poisson varieties:

$$\left(\mathrm{GL}_n, \{\,,\}_{(\boldsymbol{\Gamma}^r,\boldsymbol{\Gamma}^c)}\right) \ni X \mapsto (X,X) \in \left(D(\mathrm{GL}_n), \{\,,\}^D_{(\boldsymbol{\Gamma}^r,\boldsymbol{\Gamma}^c)}\right);$$

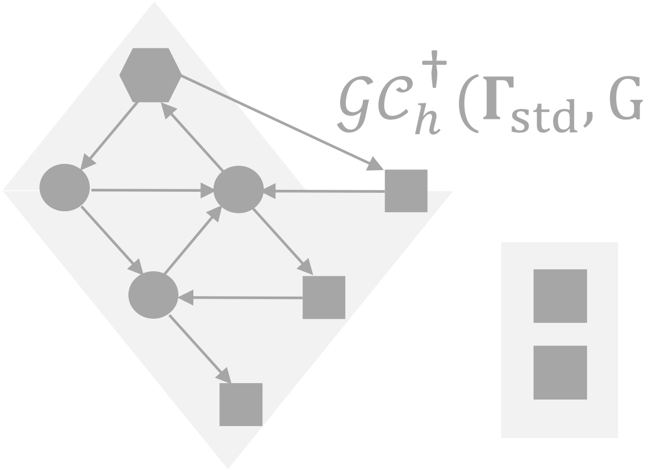
$$\left(D(\mathrm{GL}_n), \{\,,\}^D_{(\boldsymbol{\Gamma}^r,\boldsymbol{\Gamma}^c)}\right) \ni (X,Y) \mapsto U := X^{-1}Y \in \left(\mathrm{GL}_n, \{\,,\}^{\dagger}_{\boldsymbol{\Gamma}^c}\right).$$

$D(\mathrm{GL}_n) := \mathrm{GL}_n \times \mathrm{GL}_n$ (as a Lie group)



$\mathcal{GC}(\boldsymbol{\Gamma}_{\mathrm{std}}, \boldsymbol{\Gamma}_{\mathrm{std}})$

GL_3



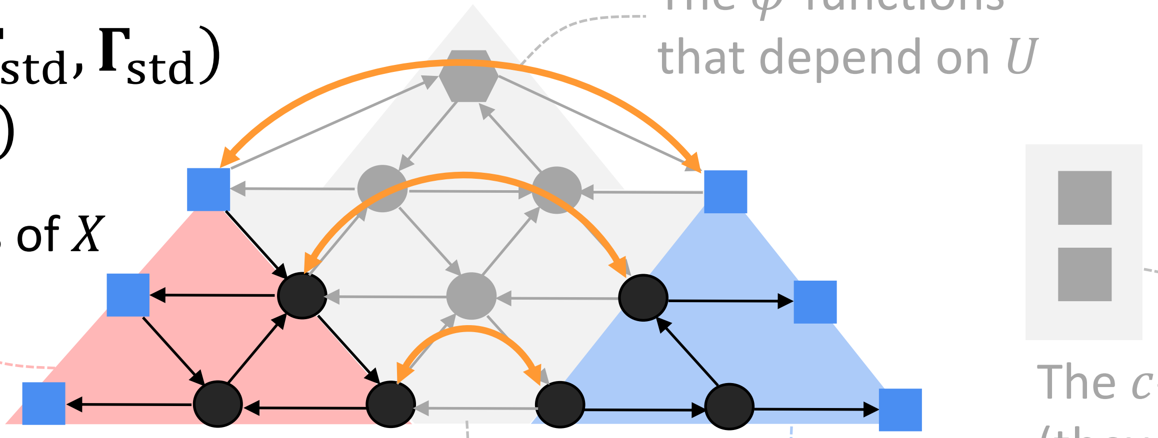
$\mathcal{GC}_h^{\dagger}(\boldsymbol{\Gamma}_{\mathrm{std}}, \mathrm{GL}_3)$

$\mathcal{GC}^D(\boldsymbol{\Gamma}_{\mathrm{std}}, \boldsymbol{\Gamma}_{\mathrm{std}})$

$\mathrm{D}(\mathrm{GL}_3)$

The φ-functions that depend on U

Flag minors of X



The c-functions (they depend on U)

Minors of both X and Y

Flag minors of Y

Coordinates $(X,Y) \in D(\mathrm{GL}_n)$

Set $X = Y$ and merge the triangles on the sides.

How to see $\mathcal{GC}_h^\dagger$ inside $\mathcal{GC}^D$

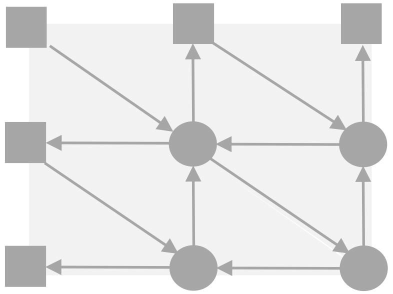
(confirmed only for $n \in \{3,4\}$ and all $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$)

Relation between the three Poisson varieties:

$\left(\mathrm{GL}_n, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right) \ni X \mapsto (X, X) \in \left(D(\mathrm{GL}_n), \{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right);$

$\left(D(\mathrm{GL}_n), \{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right) \ni (X, Y) \mapsto U := X^{-1}Y \in (\mathrm{GL}_n, \{\,,\}^\dagger_{\mathbf{\Gamma}^c}).$

$D(\mathrm{GL}_n) := \mathrm{GL}_n \times \mathrm{GL}_n$ (as a Lie group)

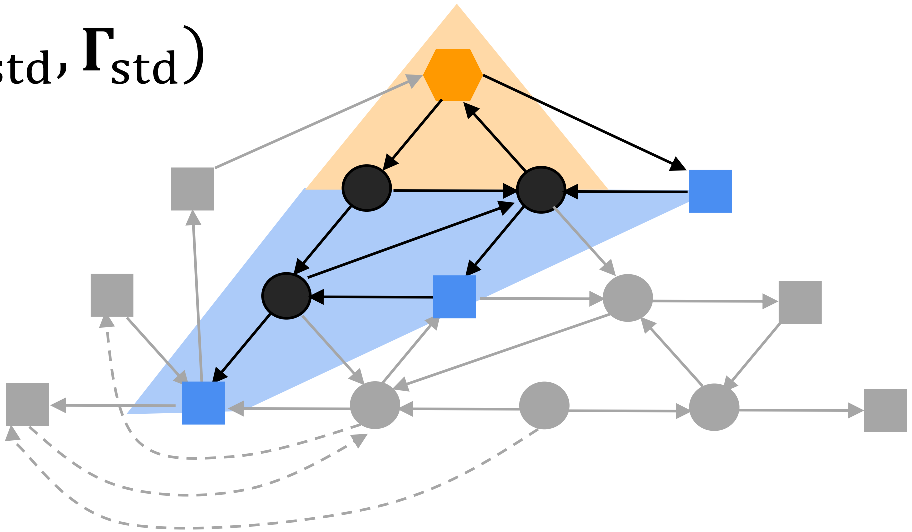


$\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

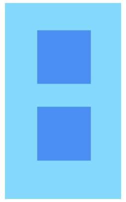
GL_3

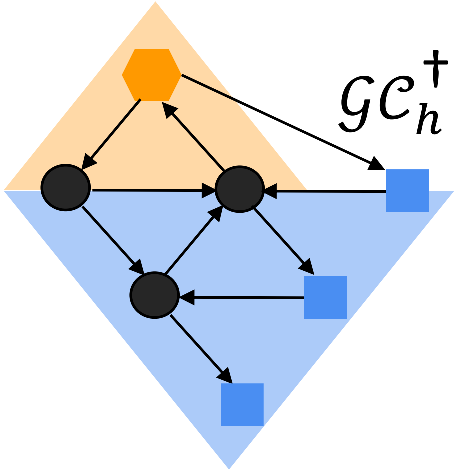
$\mathcal{GC}^D(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$

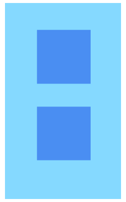
$\mathrm{D}(\mathrm{GL}_3)$



$\mathcal{GC}_h^\dagger(\mathbf{\Gamma}_{\mathrm{std}}, \mathrm{GL}_3)$







There is a mutation sequence from the initial extended seed of $\mathcal{GC}^D$ that yields the initial extended seed of $\mathcal{GC}_h^\dagger$.

Status of the GSV program

Setup. G is a connected simply connected simple complex Lie group.
(nothing is known for non-simply connected groups)

Trivial BD data:

Case of $\mathcal{GC}$	$\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$	$\mathcal{GC}^D(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$	$\mathcal{GC}^\dagger(\mathbf{\Gamma}_{\mathrm{std}})$
What we know	All Lie types	Only type A	Only type A

Problem: what to replace the φ-functions with?

Nontrivial BD data:

Case of $\mathcal{GC}$	$\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$	$\mathcal{GC}^D(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$	$\mathcal{GC}^\dagger(\mathbf{\Gamma}^c)$
What we know	All Lie types, aperiodic $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$	Only type A, aperiodic $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$	Only type A, all $\mathbf{\Gamma}^c$

(for $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$, only type A is published)

Problem: $\mathcal{U}^{-1}$ is not rational when $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ is not aperiodic.

Thank you!

List of all Poisson quasi-isomorphisms for the GSV program

(any topology of G, any Lie type, any BD pair $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$)

Gauss decomposition:

For a generic $X \in \mathrm{GL}_n$, write $X = X_+X_0X_-$ where $X_\pm \in \mathcal{N}_\pm$ and $X_0 \in \mathcal{H}$.

Recall the maps $\rho_r\colon \mathcal{N}_+ \to \mathcal{N}_+$ and $\rho_c^*\colon \mathcal{N}_- \to \mathcal{N}_-$:

$$\rho_r(N_+) := \overleftarrow{\prod_{i\geq 1}} (\tilde{\gamma}_r)^i(N_+), \quad \rho_c^*(N_-) := \overrightarrow{\prod_{j\geq 1}} (\tilde{\gamma}_c^*)^j(N_-), \qquad N_\pm \in \mathcal{N}_\pm.$$

The map $\mathcal{U}^D\colon \left(D(G), \{\,,\}^D_{\mathrm{std}}\right) \dashrightarrow \left(D(G), \{\,,\}^D_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)}\right)$ for the Drinfeld double:

$$\mathcal{U}^D(X,Y) := \left(\rho_r([W_0X]_+)X\rho_c^*([YW_0]_-), \rho_r([W_0X]_+)Y\rho_c^*([YW_0]_-)\right), \quad (X,Y) \in D(G).$$

The map $\mathcal{U}\colon (G, \{\,,\}_{\mathrm{std}}) \dashrightarrow (G, \{\,,\}_{(\mathbf{\Gamma}^r,\mathbf{\Gamma}^c)})$: $\mathcal{U}(X) := \mathcal{U}^D(X,X)$.

The maps $\mathcal{Q}_h, \mathcal{Q}_g\colon (G, \{\,,\}^\dagger_{\mathrm{std}}) \dashrightarrow (G, \{\,,\}^\dagger_{\mathbf{\Gamma}})$ for the dagger:

$$\mathcal{Q}_h(U) := \rho^*(U_-)^{-1}U\rho^*(U_-), \;\; \mathcal{Q}_g(U_+) := \rho(U_+)U\rho(U_+)^{-1}, \qquad U \in G.$$

Latest references on the program

D. Voloshyn, M. Gekhtman, 'Generalized cluster structures on $\mathrm{SL}_n^\dagger$', Preprint, 2023. arXiv:2312.04859.

This is the latest paper on the subject with up-to-date techniques. There we constructed the Poisson birational maps $\mathcal{Q}_h$ and $\mathcal{Q}_g$ along with many others, and we constructed $\mathcal{GC}^\dagger(\mathbf{\Gamma})$ for all BD triples $\mathbf{\Gamma}$ on SL_n. However, we don't know how to construct $\mathcal{GC}^\dagger(\mathbf{\Gamma}_{\mathrm{std}})$ for other Lie types. Once we know that, by means of $\mathcal{Q}_h$ or $\mathcal{Q}_g$ we will be able to construct $\mathcal{GC}^\dagger(\mathbf{\Gamma})$ for all BD triples $\mathbf{\Gamma}$.

D. Voloshyn, 'Starfish lemma via birational quasi-isomorphisms', Preprint, 2023. arXiv:2311.00404.

This paper contains techniques for proving that the upper cluster algebra equals the coordinate ring, as well as for proving irreducibility and coprimality of variables.

D. Voloshyn, github repository: github.com/Grabovskii/Dagger_examples

Here, one can find supplementary information on $\mathcal{GC}^\dagger(\mathbf{\Gamma})$, as well as explicitly computed examples.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'A unified approach to exotic cluster structures on simple Lie groups', Preprint, 2023. arXiv:2308.16701.

The paper offers a construction of a Poisson quasi-isomorphism $\boldsymbol{h}\colon (G, \{\ ,\ \}_{\mathrm{std}}) \dashrightarrow (G, \{\ ,\ \}_{(\boldsymbol{\Gamma}^r, \boldsymbol{\Gamma}^c)})$. However, by now we know a simple explicit formula (see the slides), and $\mathcal{U} = \boldsymbol{h}$. The paper also contains a combinatorial description of the initial extended seed for SL_n when the BD pair is aperiodic but not necessarily oriented.

D. Voloshyn, 'Multiple generalized cluster structures on $D(\mathrm{GL}_n)$', Forum of Mathematics, Sigma (11)(46) (2023), 1–78. doi:10.1017/fms.2023.44

By now, we know that the results of this paper could be obtained in a simpler and more conceptual way, via the Poisson birational quasi-isomorphism $\mathcal{U}^D$; for the proof of the equality between the upper cluster algebra and the coordinate ring, one can now use the results of the paper 'Starfish lemma via birational quasi-isomorphisms'.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Periodic staircase matrices and generalized cluster structures', Int. Math. Res. Not. IMRN 2022(6), 4181-4221. arXiv:1912.00453.

This paper studies generalized patterns of cluster mutations, and it presents an ad hoc way of generating such patterns. Then (in type A) the authors use them for constructing generalized cluster structures compatible with non-aperiodic BD pairs. However, it is not clear how it can be generalized to other Lie types, and so far there are no tools for proving that the procedure always works.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Drinfeld double of GL_n and generalized cluster structures', Proc. Lond. Math. Soc. 116(3) (2018), 429-484. arXiv:1605.05705.

This is the first paper in which $\mathcal{GC}^D(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$ and $\mathcal{GC}^\dagger(\mathbf{\Gamma}_{\mathrm{std}})$ were constructed (in type A). Also, it was shown how $\mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$, $\mathcal{GC}^D(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$ and $\mathcal{GC}^\dagger(\mathbf{\Gamma}_{\mathrm{std}})$ are related to one another. The paper preserves its actuality, for it contains a proof of $\mathcal{O} = \bar{\mathcal{A}}$ in the case of the trivial BD triples.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Generalized cluster structures related to the Drinfeld double of GL_n', J. Lond. Math. Soc. 105(3) (2022), 1601-1633. arXiv:2004.05118.

A companion paper to 'Periodic staircase matrices and generalized cluster mutations'.

Some older references on the program

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Plethora of cluster structures on GL_n', Preprint, 2019. arXiv:1902.02902.

This was the first paper in the GSV program where a birational quasi-isomorphism was used, but in a limited context: only to simplify the proof that the upper cluster algebra the coordinate ring. The methods are already obsolete, but the paper contains a thorough combinatorial description of the initial extended seed on SL_n when $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ is aperiodic and oriented.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Cluster algebras and Poisson geometry', Mosc. Math. J. 3(3) (2003), 899-934; arXiv:0208033.

The first paper on the connection between cluster algebras and Poisson geometry. In particular, the compatibility equation is formulated in a more general way than in the presentation.

M. Gekhtman, M. Shapiro and A. Vainshtein, 'Cluster structures on simple complex Lie groups and Belavin-Drinfeld classification', Mosc. Math. J. 12(2) (2010), 293-312; arXiv:1101.0015.

The paper that initiated the research program on constructing cluster structures compatible with Belavin-Drinfeld Poisson brackets in the coordinate rings of simple simply connected complex algebraic groups.

Other references mentioned in slides

S. Fomin and A. Zelevinsky, ’Cluster algebras I: Foundations’, J. Amer. Math. Soc. 15(2) (2002), 497–529. doi:10.1090/S0894-0347-01-00385-X

Classics. The first paper on cluster algebras, where the main definitions were formulated and a research program on cluster algebras was suggested.

B. Nguyen, K. Trampel and M. Yakimov, ’Noncommutative discriminants via Poisson primes’, Adv. Math. 322(2017), 269-307. doi:10.1016/j.aim.2017.10.018

In this paper in Remark 2.4 one can find a geometric interpretation (with a proof) of Poisson prime ideals.