Examples of generalized cluster structures on $D(\mathrm{GL}_n)$

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Abstract

This is a supplementary note for the Gekhtman-Shapiro-Vainshtein conjecture that consists of explicit examples of generalized cluster structures on the Drinfeld double $D(\mathrm{GL}_n)$ of GL_n. More examples will be added over time.

Contents

1 Introduction

The general construction of generalized cluster structures $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ on $D(\mathrm{GL}_n)$ for different Belavin-Drinfeld pairs $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ is described in our main paper [1]. Let us recall that the initial extended cluster $\Psi_0(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ of each such $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ comprises five types of variables, which are g-, h-, φ-, f- and c-variables. Only the description of g- and h-variables depends on the choice of $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$. As elements of $\mathcal{O}(D(\mathrm{GL}_n))$, these are constructed as trailing minors of the so-called $\mathcal{L}$-matrices. In the examples below, we provide the initial quiver, the list of $\mathcal{L}$-matrices and some examples of birational quasi-isomorphisms (see also [2] for definitions). The marker for related pairs $(\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c), \mathcal{GC}(\tilde{\mathbf{\Gamma}}^r, \tilde{\mathbf{\Gamma}}^c))$ is always chosen in such a way that a variable in $\Psi_0(\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c))$ corresponds to the variable with the same name and indices in $\Psi_0(\mathcal{GC}(\tilde{\mathbf{\Gamma}}^r, \tilde{\mathbf{\Gamma}}^c))$ (for instance, g_{32} in $\Psi_0(\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c))$ is related to g_{32} in $\Psi_0(\mathcal{GC}(\tilde{\mathbf{\Gamma}}^r, \tilde{\mathbf{\Gamma}}^c))$). The trivial BD triple is denoted as $\mathbf{\Gamma}_{\text{std}}$.

Conventions. For a matrix A of size $m \times n$ and subsets $I \subseteq [1, m]$, $J \subseteq [1, n]$, A_I^J denotes a submatrix of A with rows given by I and columns given by J. The standard coordinates on $D(\mathrm{GL}_n) = \mathrm{GL}_n \times \mathrm{GL}_n$ will be denoted as (X, Y). We also set $U := X^{-1}Y$.

2 Examples in $n = 3$

In this section, we list examples of $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ for some of the BD pairs $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ in $n = 3$. The φ-, f- and c-variables are given by the following formulas:

$$\varphi_{11}(X,Y) = \det X^2 \left(u_{23} \det U^{[2,3]}_{[1,2]} + u_{13} \det U^{\{1,3\}}_{[1,2]}\right); \tag{2.1}$$

$$\varphi_{12}(X,Y) = \det \begin{bmatrix} X^{\{3\}} & Y^{[2,3]} \end{bmatrix}, \quad \varphi_{21}(X,Y) = \det \begin{bmatrix} X^{[2,3]} & Y^{\{3\}} \end{bmatrix}; \tag{2.2}$$

$$f_{11}(X,Y) = \det \begin{bmatrix} X^{\{3\}} & Y^{\{3\}} \end{bmatrix}_{[2,3]}; \tag{2.3}$$

$$c_1(X,Y) = \det X \cdot \operatorname{tr}(U), \quad c_2(X,Y) = \frac{\det X}{2!}(\operatorname{tr}(U)^2 - \operatorname{tr}(U^2)). \tag{2.4}$$

2.1 Case of $\mathbf{\Gamma}^r = \mathbf{\Gamma}^c = \mathbf{\Gamma}_{\mathbf{std}}$

The initial quiver is depicted in Figure 1.

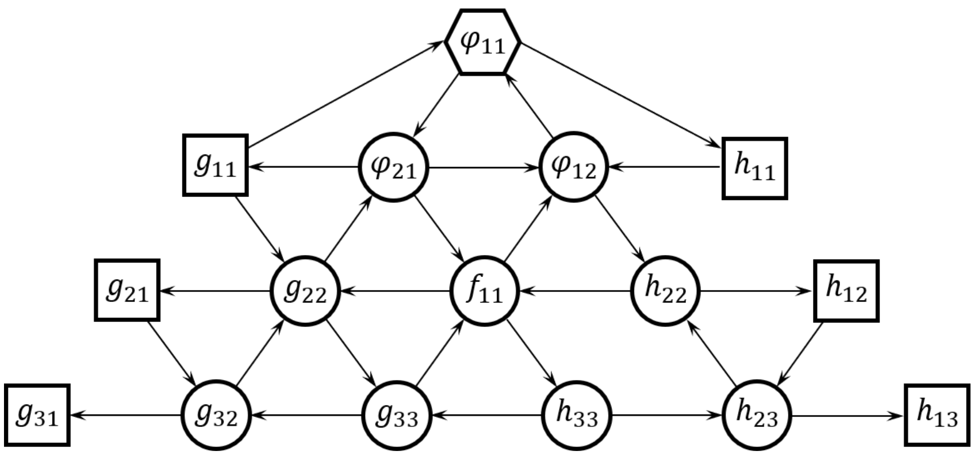

Figure 1. *The initial quiver for $\mathcal{GC}(\mathbf{\Gamma}_{std}, \mathbf{\Gamma}_{std})$ on $D(\mathrm{GL}_3)$.* {f:ex_n=3}

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = x_{31}, \quad \mathcal{L}_2(X,Y) = X^{[1,2]}_{[2,3]}, \quad \mathcal{L}_3(X,Y) = y_{13}, \quad \mathcal{L}_4(X,Y) = Y^{[2,3]}_{[1,2]}. \tag{2.5}$$

2.2 Case of $\mathbf{\Gamma}^r = (\{1\},\{2\},1\mapsto 2)$, $\mathbf{\Gamma}^c = \mathbf{\Gamma}_{\mathbf{std}}$

The initial quiver is depicted in Figure 2.

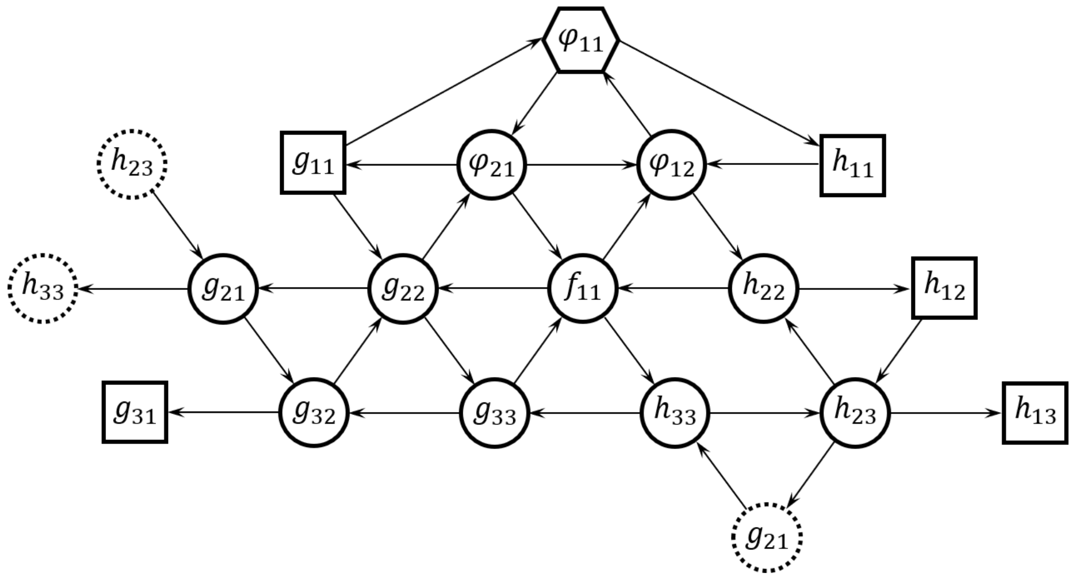

Figure 2. *The initial quiver for* $\mathbf{\Gamma}^r = (\{1\},\{2\},1\mapsto 2)$, $\mathbf{\Gamma}^c = \mathbf{\Gamma}_{std}$. {f:ex_n=3_r1

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = x_{31}, \quad \mathcal{L}_2(X,Y) = \begin{bmatrix} y_{12} & y_{13} & 0 & 0 \\ y_{22} & y_{23} & x_{11} & x_{12} \\ y_{32} & y_{33} & x_{21} & x_{22} \\ 0 & 0 & x_{31} & x_{32} \end{bmatrix}, \quad \mathcal{L}_3(X,Y) = y_{13}. \tag{2.6}$$

Birational quasi-isomorphism. Define

$$U(X,Y) := I + \frac{\det X^{[1,2]}_{\{1,3\}}}{\det X^{[1,2]}_{[2,3]}} e_{23}. \tag{2.7}$$

There is a birational quasi-isomorphism $\mathcal{U}^* : \mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}_{\mathrm{std}}) \to \mathcal{GC}(\mathbf{\Gamma}_{\mathrm{std}}, \mathbf{\Gamma}_{\mathrm{std}})$ given by

$$\mathcal{U}(X,Y) = (U(X,Y)\cdot X, U(X,Y)\cdot Y). \tag{2.8}$$

The marked variable for the related pair is g_{21}.

2.3 Case of $\mathbf{\Gamma}^r = \mathbf{\Gamma}_{\mathbf{std}}$, $\mathbf{\Gamma}^c = (\{1\},\{2\},1\mapsto 2)$

The initial quiver is depicted in Figure 3.

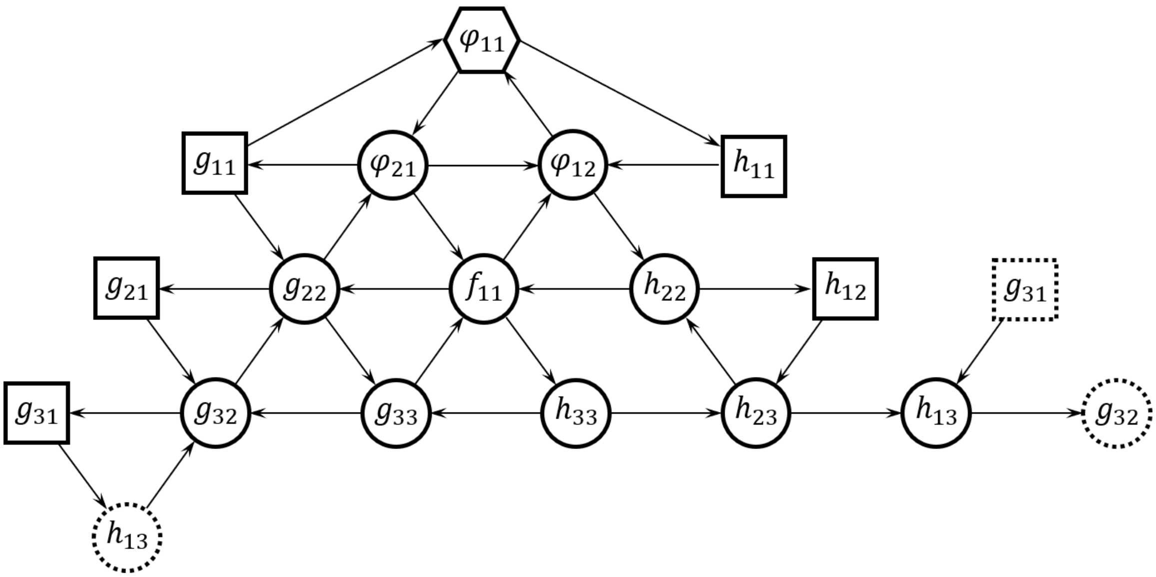

Figure 3. *The initial quiver for* $\mathbf{\Gamma}^r = \mathbf{\Gamma}_{std}$, $\mathbf{\Gamma}^c = (\{1\},\{2\},1\mapsto 2)$. {f:ex_n=3_c1

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = X^{[1,2]}_{[2,3]}, \quad \mathcal{L}_2(X,Y) = \begin{bmatrix} x_{31} & x_{32} \\ y_{12} & y_{13} \end{bmatrix}, \quad \mathcal{L}_3(X,Y) = Y^{[2,3]}_{[1,2]}. \tag{2.9}$$

Birational quasi-isomorphism. Define

$$U(X,Y) := I + \frac{y_{12}}{y_{13}} e_{21}. \tag{2.10}$$

There is a birational quasi-isomorphism $\mathcal{U}^* : \mathcal{GC}(\mathbf{\Gamma}_{\rm std}, \mathbf{\Gamma}^c) \to \mathcal{GC}(\mathbf{\Gamma}_{\rm std}, \mathbf{\Gamma}_{\rm std})$ given by

$$\mathcal{U}(X,Y) = (X \cdot U(X,Y), Y \cdot U(X,Y)). \tag{2.11}$$

The marked variable for the related pair is h_{13}.

2.4 Case of $\mathbf{\Gamma}^r = \mathbf{\Gamma}^c = (\{1\}, \{2\}, 1 \mapsto 2)$

The initial quiver is depicted in Figure 4.

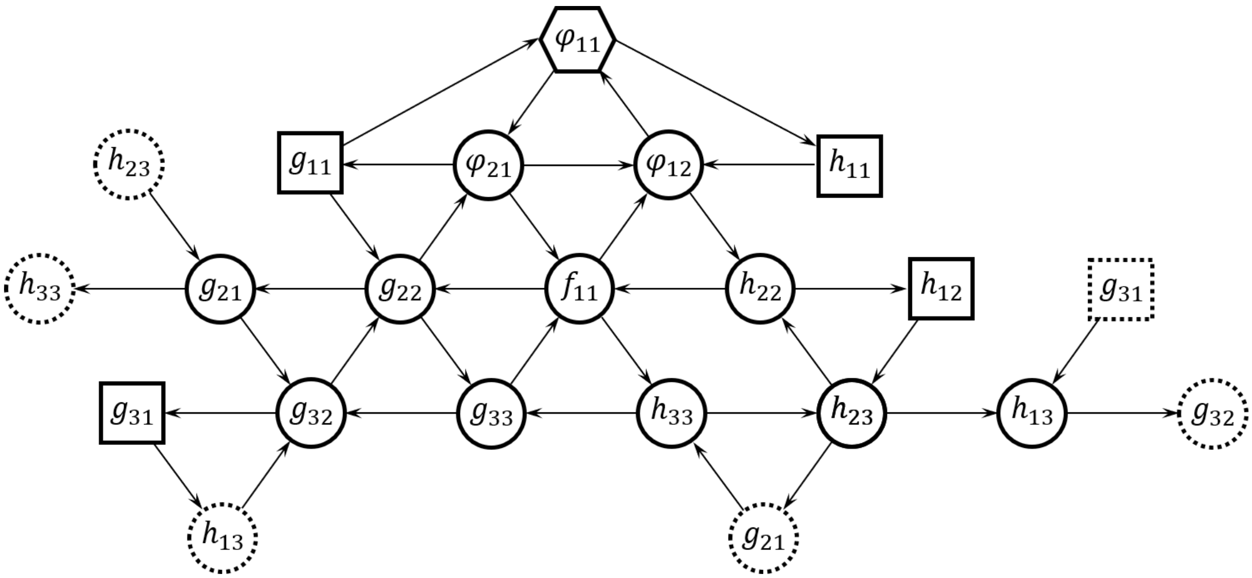

Figure 4. *The initial quiver for* $\mathbf{\Gamma}^r = \mathbf{\Gamma}^c = (\{1\}, \{2\}, 1 \mapsto 2)$. {f:ex_n=3_rc

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = \begin{bmatrix} y_{12} & y_{13} & 0 & 0 \\ y_{22} & y_{23} & x_{11} & x_{12} \\ y_{32} & y_{33} & x_{21} & x_{22} \\ 0 & 0 & x_{31} & x_{32} \end{bmatrix}, \quad \mathcal{L}_2(X,Y) = \begin{bmatrix} x_{31} & x_{32} \\ y_{12} & y_{13} \end{bmatrix}. \tag{2.12}$$

Birational quasi-isomorphism to $\mathcal{GC}(\mathbf{\Gamma_{std}}, \mathbf{\Gamma_{std}})$. Define

$$U_r(X,Y) := I + \frac{\det X_{\{1,3\}}^{[1,2]}}{\det X_{[2,3]}^{[1,2]}} e_{23}; \tag{2.13}$$

$$U_c(X,Y) := I + \frac{y_{12}}{y_{13}} e_{21}. \tag{2.14}$$

There is a birational quasi-isomorphism $\mathcal{U}^* : \mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c) \to \mathcal{GC}(\mathbf{\Gamma}_{\text{std}}, \mathbf{\Gamma}_{\text{std}})$ given by

$$\mathcal{U}(X,Y) = (U_r(X,Y) \cdot X \cdot U_c(X,Y), U_r(X,Y) \cdot Y \cdot U_c(X,Y)). \tag{2.15}$$

The marked variables for the related pair are g_{21} and h_{13}. There is also a pair of complementary birational quasi-isomorphisms $\mathcal{U}_c : \mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c) \to \mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}_{\text{std}})$ and $\mathcal{U}_r : \mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c) \to \mathcal{GC}(\mathbf{\Gamma}_{\text{std}}, \mathbf{\Gamma}^c)$ which are given by

$$\mathcal{U}_c(X,Y) = (X \cdot U_c(X,Y), Y \cdot U_c(X,Y)); \tag{2.16}$$

$$\mathcal{U}_r(X,Y) = (U_r(X,Y) \cdot X, U_r(X,Y) \cdot Y). \tag{2.17}$$

We also see that $\mathcal{U} = \mathcal{U}_r \circ \mathcal{U}_c = \mathcal{U}_c \circ \mathcal{U}_r$.

3 Examples in $n = 4$

In this section, we list examples of $\mathcal{GC}(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ for some of the BD pairs $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ in $n = 4$.

3.1 Case of $\mathbf{\Gamma}^r = \mathbf{\Gamma}^c = \mathbf{\Gamma}_{\mathbf{std}}$

The initial quiver is depicted in Figure 5.

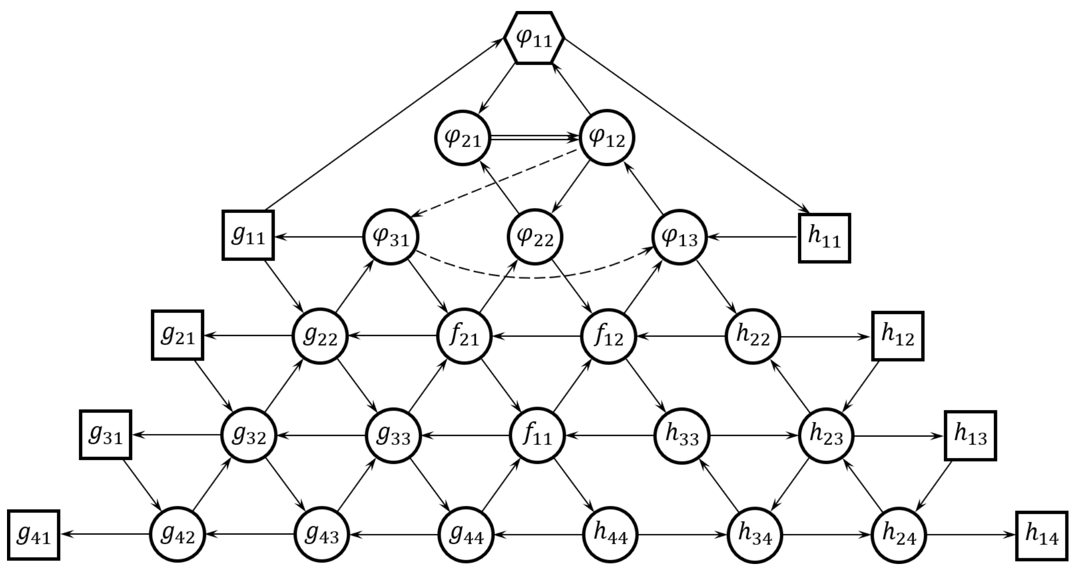

Figure 5. *The initial quiver for* $\mathcal{GC}(\mathbf{\Gamma}_{std}, \mathbf{\Gamma}_{std})$ *on* $D(\mathrm{GL}_4)$. {f:ex_n=4}

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = x_{41}, \quad \mathcal{L}_2(X,Y) = X^{[1,2]}_{[3,4]}, \quad \mathcal{L}_3(X,Y) = X^{[1,3]}_{[2,4]}; \tag{3.1}$$

$$\mathcal{L}_4(X,Y) = y_{14}, \quad \mathcal{L}_5(X,Y) = Y^{[3,4]}_{[1,2]}, \quad \mathcal{L}_6(X,Y) = Y^{[2,4]}_{[1,3]}. \tag{3.2}$$

3.2 Case of Cremmer-Gervais, $i \mapsto i+1$

The initial quiver is depicted in Figure 6.

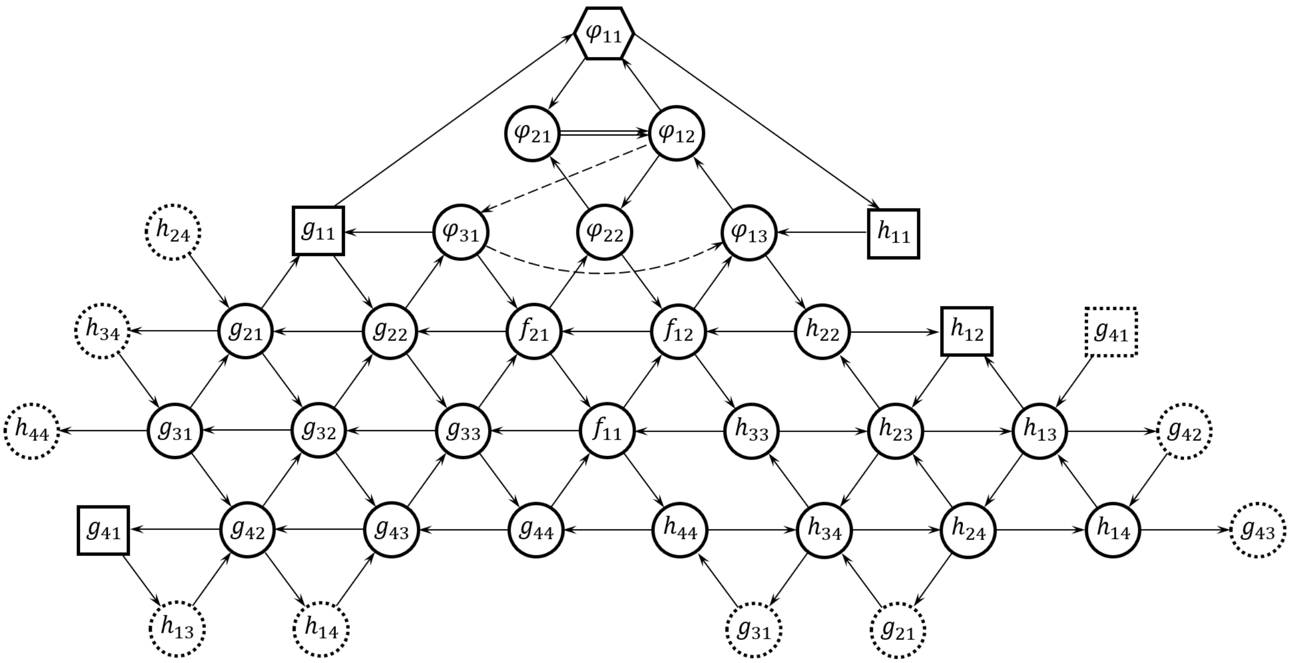

Figure 6. *The initial quiver for* $\mathbf{\Gamma}^r = \mathbf{\Gamma}^c = (\{1,2\},\{2,3\}, i \mapsto i+1)$. {f:n=4_rc12_

The $\mathcal{L}$-matrices. These are given by:

$$\mathcal{L}_1(X,Y) = \begin{bmatrix} x_{41} & x_{42} & x_{43} & 0 & 0 & 0 \\ y_{12} & y_{13} & y_{14} & 0 & 0 & 0 \\ y_{22} & y_{23} & y_{24} & x_{11} & x_{12} & x_{13} \\ y_{32} & y_{33} & y_{34} & x_{21} & x_{22} & x_{23} \\ y_{42} & y_{43} & y_{44} & x_{31} & x_{32} & x_{33} \\ 0 & 0 & 0 & x_{41} & x_{42} & x_{43} \end{bmatrix}, \quad \mathcal{L}_2(X,Y) = \begin{bmatrix} y_{12} & y_{13} & y_{14} & 0 & 0 & 0 \\ y_{22} & y_{23} & y_{24} & x_{11} & x_{12} & x_{13} \\ y_{32} & y_{33} & y_{34} & x_{21} & x_{22} & x_{23} \\ y_{42} & y_{43} & y_{44} & x_{31} & x_{32} & x_{33} \\ 0 & 0 & 0 & x_{41} & x_{42} & x_{43} \\ 0 & 0 & 0 & y_{12} & y_{13} & y_{14} \end{bmatrix}. \qquad (3.3)$$

References

[1] D. Voloshyn, 'Multiple generalized cluster structures on $D(\mathrm{GL}_n)$', *Forum of Mathematics, Sigma* (**11**)(46) (2023), 1–78. doi:10.1017/fms.2023.44

[2] D. Voloshyn, 'Note on birational quasi- isomorphisms', Preprint, 2023, *to appear*.